Manin matrices, Casimir elements and Sugawara operators

Alexander Molev

University of Sydney

Plan

Plan

- Origins and motivations.

Plan

- Origins and motivations.
- Basic properties of Manin matrices.

Plan

- Origins and motivations.
- Basic properties of Manin matrices.
- Applications:
  - Casimir elements for $\mathfrak{gl}_n$.

Plan

- Origins and motivations.
- Basic properties of Manin matrices.
- Applications:
  - Casimir elements for $\mathfrak{gl}_n$.
  - Segal–Sugawara vectors for $\mathfrak{gl}_n$.

Quantum groups

Quantum groups

By the seminal works of V. Drinfeld (1985) and M. Jimbo (1985), the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$ admits a deformation $\mathrm{U}_q(\mathfrak{g})$ in the class of Hopf algebras.

Quantum groups

By the seminal works of V. Drinfeld (1985) and M. Jimbo (1985),

the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$

admits a deformation $\mathrm{U}_q(\mathfrak{g})$ in the class of Hopf algebras.

The dual Hopf algebras are quantized algebras of functions

$\mathrm{Fun}_q(G)$ on the associated Lie group G

[N. Reshetikhin, L. Takhtajan and L. Faddeev 1990].

Quantum groups

By the seminal works of V. Drinfeld (1985) and M. Jimbo (1985), the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$ admits a deformation $\mathrm{U}_q(\mathfrak{g})$ in the class of Hopf algebras.

The dual Hopf algebras are quantized algebras of functions $\mathrm{Fun}_q(G)$ on the associated Lie group G [N. Reshetikhin, L. Takhtajan and L. Faddeev 1990].

A detailed review of the theory and applications:

V. Chari and A. Pressley, *A guide to quantum groups*, 1994.

Basic example

Basic example

The algebra $\mathrm{Fun}_q(\mathrm{Mat}_2)$ is generated by four elements a, b, c, d,

understood as the entries of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, such that

Basic example

The algebra $\mathrm{Fun}_q(\mathrm{Mat}_2)$ is generated by four elements a, b, c, d, understood as the entries of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, such that

$$ba = qab, \qquad dc = qcd, \qquad ca = qac, \qquad db = qbd,$$

Basic example

The algebra $\mathrm{Fun}_q(\mathrm{Mat}_2)$ is generated by four elements a, b, c, d, understood as the entries of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, such that

$$ba = qab, \qquad dc = qcd, \qquad ca = qac, \qquad db = qbd,$$

and

$$bc = cb, \qquad ad - da + (q - q^{-1})bc = 0.$$

Basic example

The algebra $\mathrm{Fun}_q(\mathrm{Mat}_2)$ is generated by four elements a, b, c, d,

understood as the entries of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, such that

$$ba = qab, \qquad dc = qcd, \qquad ca = qac, \qquad db = qbd,$$

and

$$bc = cb, \qquad ad - da + (q - q^{-1})bc = 0.$$

[L. Faddeev and L. Takhtajan 1986].

As observed by Yu. Manin (1988), the relations are recovered via a “coaction” on the quantum plane: the algebra with generators x, y and the relation $yx = qxy$.

As observed by Yu. Manin (1988), the relations are recovered via a "coaction" on the quantum plane: the algebra with generators x, y and the relation $yx = qxy$.

A 2×2 matrix is q-Manin if the elements x' and y' defined by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

satisfy $y'x' = qx'y'$.

As observed by Yu. Manin (1988), the relations are recovered via a "coaction" on the quantum plane: the algebra with generators x, y and the relation $yx = qxy$.

A 2×2 matrix is q-Manin if the elements x' and y' defined by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

satisfy $y'x' = qx'y'$. The defining relations for $\mathrm{Fun}_q(\mathrm{Mat}_2)$ are equivalent to the conditions that the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and its transpose are q-Manin matrices.

MacMahon Master Theorem

MacMahon Master Theorem

Consider the characteristic polynomial of a matrix

$$M = \begin{bmatrix} M_{11} & \dots & M_{1n} \\ \vdots & \vdots & \vdots \\ M_{n1} & \dots & M_{nn} \end{bmatrix}$$

with entries in $\mathbb{C}$:

MacMahon Master Theorem

Consider the characteristic polynomial of a matrix

$$M = \begin{bmatrix} M_{11} & \dots & M_{1n} \\ \vdots & \vdots & \vdots \\ M_{n1} & \dots & M_{nn} \end{bmatrix}$$

with entries in $\mathbb{C}$:

$$\det(I + tM) = 1 + \Delta_1 t + \cdots + \Delta_n t^n.$$

MacMahon Master Theorem

Consider the characteristic polynomial of a matrix

$$M = \begin{bmatrix} M_{11} & \dots & M_{1n} \\ \vdots & \vdots & \vdots \\ M_{n1} & \dots & M_{nn} \end{bmatrix}$$

with entries in $\mathbb{C}$:

$$\det(I + tM) = 1 + \Delta_1 t + \cdots + \Delta_n t^n.$$

In particular,

$$\Delta_1 = \operatorname{tr} M, \qquad \Delta_n = \det M.$$

Regard the matrix M as the element

$$M = \sum_{i,j=1}^{n} e_{ij} \otimes M_{ij} \in \operatorname{End} \mathbb{C}^n \otimes \mathbb{C}.$$

Regard the matrix M as the element

$$M = \sum_{i,j=1}^{n} e_{ij} \otimes M_{ij} \in \operatorname{End} \mathbb{C}^n \otimes \mathbb{C}.$$

Consider the algebra

$$\underbrace{\operatorname{End} \mathbb{C}^n \otimes \ldots \otimes \operatorname{End} \mathbb{C}^n}_{k} \otimes \mathbb{C}$$

Regard the matrix M as the element

$$M = \sum_{i,j=1}^{n} e_{ij} \otimes M_{ij} \in \operatorname{End} \mathbb{C}^n \otimes \mathbb{C}.$$

Consider the algebra

$$\underbrace{\operatorname{End} \mathbb{C}^n \otimes \ldots \otimes \operatorname{End} \mathbb{C}^n}_{k} \otimes \mathbb{C}$$

and for $a = 1, \ldots, k$ set

$$M_a = \sum_{i,j=1}^{n} \underbrace{I \otimes \ldots \otimes I}_{a-1} \otimes e_{ij} \otimes \underbrace{I \otimes \ldots \otimes I}_{k-a} \otimes M_{ij}.$$

The symmetric group $\mathfrak{S}_k$ acts on the tensor product space

$$\underbrace{\mathbb{C}^n \otimes \ldots \otimes \mathbb{C}^n}_{k}$$

by permutations of tensor factors.

The symmetric group $\mathfrak{S}_k$ acts on the tensor product space

$$\underbrace{\mathbb{C}^n \otimes \ldots \otimes \mathbb{C}^n}_{k}$$

by permutations of tensor factors.

Denote by $H^{(k)}$ and $A^{(k)}$ the respective images of the

symmetrizer and anti-symmetrizer

$$\frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \sigma \qquad \text{and} \qquad \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn} \sigma \cdot \sigma.$$

The symmetric group $\mathfrak{S}_k$ acts on the tensor product space

$$\underbrace{\mathbb{C}^n \otimes \ldots \otimes \mathbb{C}^n}_{k}$$

by permutations of tensor factors.

Denote by $H^{(k)}$ and $A^{(k)}$ the respective images of the

symmetrizer and anti-symmetrizer

$$\frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \sigma \qquad \text{and} \qquad \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn} \sigma \cdot \sigma.$$

We regard $H^{(k)}$ and $A^{(k)}$ as elements of the algebra

$$\underbrace{\operatorname{End} \mathbb{C}^n \otimes \ldots \otimes \operatorname{End} \mathbb{C}^n}_{k}.$$

We have

$$\Delta_k = \operatorname{tr} A^{(k)} M_1 \dots M_k, \qquad k = 1, \dots, n.$$

We have

$$\Delta_k = \operatorname{tr} A^{(k)} M_1 \dots M_k, \qquad k = 1, \dots, n.$$

Introduce the sums

$$\text{Ferm} = 1 + \sum_{k=1}^{n} (-1)^k \operatorname{tr} A^{(k)} M_1 \dots M_k = \det(I - M),$$

$$\text{Bos} = 1 + \sum_{k=1}^{\infty} \operatorname{tr} H^{(k)} M_1 \dots M_k.$$

We have

$$\Delta_k = \operatorname{tr} A^{(k)} M_1 \dots M_k, \qquad k = 1, \dots, n.$$

Introduce the sums

$$\text{Ferm} = 1 + \sum_{k=1}^{n} (-1)^k \operatorname{tr} A^{(k)} M_1 \dots M_k = \det(I - M),$$

$$\text{Bos} = 1 + \sum_{k=1}^{\infty} \operatorname{tr} H^{(k)} M_1 \dots M_k.$$

Theorem [P. MacMahon 1916]. We have the identity

$$\text{Bos} \times \text{Ferm} = 1.$$

Manin matrices

Manin matrices

Consider the tensor product algebra $\mathcal{A} \otimes \mathbb{C}[x, y]$.

Manin matrices

Consider the tensor product algebra $\mathcal{A} \otimes \mathbb{C}[x, y]$.

Look for 2×2 matrices over $\mathcal{A}$ such that x' and y' defined by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

commute.

Manin matrices

Consider the tensor product algebra $\mathcal{A} \otimes \mathbb{C}[x, y]$.

Look for 2×2 matrices over $\mathcal{A}$ such that x' and y' defined by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

commute. We have

$$[ax + by, cx + dy] = [a, c]\, x^2 + \big([a, d] + [b, c]\big)\, xy + [b, d]\, y^2.$$

Manin matrices

Consider the tensor product algebra $\mathcal{A} \otimes \mathbb{C}[x, y]$.

Look for 2×2 matrices over $\mathcal{A}$ such that x' and y' defined by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

commute. We have

$$[ax + by, cx + dy] = [a, c]\, x^2 + \big([a, d] + [b, c]\big)\, xy + [b, d]\, y^2.$$

This leads to the definition of Manin matrices:

$$[a, c] = [b, d] = 0 \qquad \text{and} \qquad [a, d] = [c, b].$$

Definition. An $n \times n$ matrix M over an associative algebra $\mathcal{A}$ is a Manin matrix if all its 2×2 submatrices are Manin matrices:

Definition. An $n \times n$ matrix M over an associative algebra $\mathcal{A}$ is a Manin matrix if all its 2×2 submatrices are Manin matrices:

$$\big[M_{ij}, M_{kl}\big] = \big[M_{kj}, M_{il}\big], \qquad i,j,k,l \in \{1,\dots,n\}.$$

Definition. An $n \times n$ matrix M over an associative algebra $\mathcal{A}$ is a Manin matrix if all its 2×2 submatrices are Manin matrices:

$$\big[M_{ij}, M_{kl}\big] = \big[M_{kj}, M_{il}\big], \qquad i, j, k, l \in \{1, \dots, n\}.$$

Equivalently, using the permutation operator

$$P \in \operatorname{End} \mathbb{C}^n \otimes \operatorname{End} \mathbb{C}^n, \qquad P : \xi \otimes \eta \mapsto \eta \otimes \xi$$

Definition. An $n \times n$ matrix M over an associative algebra $\mathcal{A}$ is a Manin matrix if all its 2×2 submatrices are Manin matrices:

$$\big[M_{ij}, M_{kl}\big] = \big[M_{kj}, M_{il}\big], \qquad i,j,k,l \in \{1,\dots,n\}.$$

Equivalently, using the permutation operator

$$P \in \operatorname{End}\mathbb{C}^n \otimes \operatorname{End}\mathbb{C}^n, \qquad P : \xi \otimes \eta \mapsto \eta \otimes \xi$$

we find that M is a Manin matrix if and only if

$$(1-P)\, M_1 M_2\, (1+P) = 0$$

in the algebra $\operatorname{End}\mathbb{C}^n \otimes \operatorname{End}\mathbb{C}^n \otimes \mathcal{A}$.

For any $n \times n$ matrix M over an associative algebra set

$$\mathrm{Ferm} = 1 + \sum_{k=1}^{n} (-1)^k \,\mathrm{tr}\, A^{(k)} M_1 \ldots M_k,$$

$$\mathrm{Bos} = 1 + \sum_{k=1}^{\infty} \mathrm{tr}\, H^{(k)} M_1 \ldots M_k.$$

For any $n \times n$ matrix M over an associative algebra set

$$\mathrm{Ferm} = 1 + \sum_{k=1}^{n} (-1)^k \operatorname{tr} A^{(k)} M_1 \dots M_k,$$

$$\mathrm{Bos} = 1 + \sum_{k=1}^{\infty} \operatorname{tr} H^{(k)} M_1 \dots M_k.$$

Theorem [Garoufalidis–Lê–Zeilberger 2006].

If M is a Manin matrix, then

$$\mathrm{Bos} \times \mathrm{Ferm} = 1.$$

Introduce the column-determinant of a matrix M by

$$\operatorname{cdet} M = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn} \sigma \cdot M_{\sigma(1)\,1} \dots M_{\sigma(n)\,n}.$$

Introduce the column-determinant of a matrix M by

$$\operatorname{cdet} M = \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}\sigma \cdot M_{\sigma(1)\,1} \dots M_{\sigma(n)\,n}.$$

Proposition. If M is a Manin matrix, then

$$\operatorname{cdet}(I + tM) = \sum_{k=0}^{n} t^k \operatorname{tr} A^{(k)} M_1 \dots M_k,$$

$$\big[\operatorname{cdet}(I - tM)\big]^{-1} = \sum_{k=0}^{\infty} t^k \operatorname{tr} H^{(k)} M_1 \dots M_k.$$

Introduce the column-determinant of a matrix M by

$$\operatorname{cdet} M = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn} \sigma \cdot M_{\sigma(1)\,1} \dots M_{\sigma(n)\,n}.$$

Proposition. If M is a Manin matrix, then

$$\operatorname{cdet}(I + tM) = \sum_{k=0}^{n} t^k \operatorname{tr} A^{(k)} M_1 \dots M_k,$$

$$\big[\operatorname{cdet}(I - tM)\big]^{-1} = \sum_{k=0}^{\infty} t^k \operatorname{tr} H^{(k)} M_1 \dots M_k.$$

Proof. For any $\sigma \in \mathfrak{S}_k$ we have

$$A^{(k)} M_1 \dots M_k = \operatorname{sgn} \sigma \cdot A^{(k)} M_1 \dots M_k \, P_\sigma.$$

- The Newton identity holds:

$$\frac{d}{dt}\operatorname{cdet}(I+tM)=\operatorname{cdet}(I+tM)\sum_{k=0}^{\infty}(-t)^k\operatorname{tr}M^{k+1}.$$

- The Newton identity holds:

$$\frac{d}{dt}\,\mathrm{cdet}(I+tM)=\mathrm{cdet}(I+tM)\,\sum_{k=0}^{\infty}(-t)^k\,\mathrm{tr}\,M^{k+1}.$$

- Set $C(u)=\mathrm{cdet}(uI-M)=u^n-\Delta_1\,u^{n-1}+\dots+(-1)^n\Delta_n$.

- The Newton identity holds:

$$\frac{d}{dt}\,\mathrm{cdet}(I + tM) = \mathrm{cdet}(I + tM)\sum_{k=0}^{\infty}(-t)^k\,\mathrm{tr}\,M^{k+1}.$$

- Set $C(u) = \mathrm{cdet}(uI - M) = u^n - \Delta_1 u^{n-1} + \cdots + (-1)^n \Delta_n$.

  The Cayley–Hamilton identity holds: $C(M) = 0$.

- The Newton identity holds:

$$\frac{d}{dt}\,\mathrm{cdet}(I+tM) = \mathrm{cdet}(I+tM)\sum_{k=0}^{\infty}(-t)^k\,\mathrm{tr}\,M^{k+1}.$$

- Set $C(u) = \mathrm{cdet}(uI - M) = u^n - \Delta_1 u^{n-1} + \cdots + (-1)^n \Delta_n$.

  The Cayley–Hamilton identity holds: $C(M) = 0$.

- If M and $\mathrm{cdet}\,M$ are invertible, them M^{-1} is a Manin matrix.

  Moreover, $\mathrm{cdet}\,(M^{-1}) = (\mathrm{cdet}\,M)^{-1}$.

- The Newton identity holds:

$$\frac{d}{dt}\,\mathrm{cdet}(I + tM) = \mathrm{cdet}(I + tM)\,\sum_{k=0}^{\infty}(-t)^k\,\mathrm{tr}\,M^{k+1}.$$

- Set $C(u) = \mathrm{cdet}(uI - M) = u^n - \Delta_1\,u^{n-1} + \cdots + (-1)^n\Delta_n$.

  The Cayley–Hamilton identity holds: $C(M) = 0$.

- If M and $\mathrm{cdet}\,M$ are invertible, them M^{-1} is a Manin matrix.

  Moreover, $\mathrm{cdet}\,(M^{-1}) = (\mathrm{cdet}\,M)^{-1}$.

Further properties and generalizations: [Chervov, Falqui, Foata, Han, M., Ragoucy, Rubtsov, Silantyev, … 2007–2020].

Applications: Casimir elements

Applications: Casimir elements

The Lie algebra $\mathfrak{gl}_n$ is the vector space $\mathrm{End}\,\mathbb{C}^n$ with the bracket

$$[A, B] = AB - BA.$$

Applications: Casimir elements

The Lie algebra $\mathfrak{gl}_n$ is the vector space $\mathrm{End}\,\mathbb{C}^n$ with the bracket

$$\big[A, B\big] = AB - BA.$$

The matrix units e_{ij} form its basis with the commutation relations

$$\big[e_{ij}, e_{kl}\big] = \delta_{kj}\, e_{il} - \delta_{il}\, e_{kj}.$$

Applications: Casimir elements

The Lie algebra $\mathfrak{gl}_n$ is the vector space $\mathrm{End}\,\mathbb{C}^n$ with the bracket

$$\big[A, B\big] = AB - BA.$$

The matrix units e_{ij} form its basis with the commutation relations

$$\big[e_{ij}, e_{kl}\big] = \delta_{kj}\, e_{il} - \delta_{il}\, e_{kj}.$$

The group GL_n acts on $\mathfrak{gl}_n$ by conjugation: $X \mapsto g\,X\,g^{-1}$,

Applications: Casimir elements

The Lie algebra $\mathfrak{gl}_n$ is the vector space $\mathrm{End}\,\mathbb{C}^n$ with the bracket

$$\big[A, B\big] = AB - BA.$$

The matrix units e_{ij} form its basis with the commutation relations

$$\big[e_{ij}, e_{kl}\big] = \delta_{kj}\, e_{il} - \delta_{il}\, e_{kj}.$$

The group GL_n acts on $\mathfrak{gl}_n$ by conjugation: $X \mapsto g X g^{-1}$,

and the action extends to the symmetric algebra $\mathrm{S}(\mathfrak{gl}_n)$ which

can be viewed as the algebra of polynomials in n^2 variables E_{ij}.

Consider the matrix

$$E = \begin{bmatrix} E_{11} & \dots & E_{1n} \\ \vdots & \vdots & \vdots \\ E_{n1} & \dots & E_{nn} \end{bmatrix}$$

with entries in the symmetric algebra $\mathrm{S}(\mathfrak{gl}_n)$.

Consider the matrix

$$E = \begin{bmatrix} E_{11} & \dots & E_{1n} \\ \vdots & \vdots & \vdots \\ E_{n1} & \dots & E_{nn} \end{bmatrix}$$

with entries in the symmetric algebra $\mathrm{S}(\mathfrak{gl}_n)$.

Write

$$\det(u + E) = u^n + \Delta_1 u^{n-1} + \dots + \Delta_n.$$

Consider the matrix

$$E = \begin{bmatrix} E_{11} & \dots & E_{1n} \\ \vdots & \vdots & \vdots \\ E_{n1} & \dots & E_{nn} \end{bmatrix}$$

with entries in the symmetric algebra $\mathrm{S}(\mathfrak{gl}_n)$.

Write

$$\det(u+E) = u^n + \Delta_1 u^{n-1} + \cdots + \Delta_n.$$

We have

$$\mathrm{S}(\mathfrak{gl}_n)^{\mathrm{GL}_n} = \mathbb{C}[\Delta_1, \dots, \Delta_n].$$

The universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$ is the associative algebra with n^2 generators E_{ij} and the defining relations

$$E_{ij} E_{kl} - E_{kl} E_{ij} = \delta_{kj} E_{il} - \delta_{il} E_{kj}.$$

The universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$ is the associative algebra with n^2 generators E_{ij} and the defining relations

$$E_{ij}\,E_{kl} - E_{kl}\,E_{ij} = \delta_{kj}E_{il} - \delta_{il}E_{kj}.$$

The symmetrization map

$$\varpi : \mathrm{S}(\mathfrak{gl}_n) \xrightarrow{\sim} \mathrm{U}(\mathfrak{gl}_n),$$

The universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$ is the associative algebra with n^2 generators E_{ij} and the defining relations

$$E_{ij}\,E_{kl} - E_{kl}\,E_{ij} = \delta_{kj}E_{il} - \delta_{il}E_{kj}.$$

The symmetrization map

$$\varpi : \mathrm{S}(\mathfrak{gl}_n) \xrightarrow{\sim} \mathrm{U}(\mathfrak{gl}_n),$$

is a GL_n-module isomorphism, defined by

$$\varpi : X_1 \dots X_k \mapsto \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} X_{\sigma(1)} \dots X_{\sigma(k)}, \qquad X_i \in \mathfrak{gl}_n,$$

[Poincaré–Birkhoff–Witt Theorem].

This implies the isomorphism

$$\mathrm{S}(\mathfrak{gl}_n)^{\mathrm{GL}_n} \cong \mathrm{Z}(\mathfrak{gl}_n),$$

where $\mathrm{Z}(\mathfrak{gl}_n)$ is the center of $\mathrm{U}(\mathfrak{gl}_n)$.

This implies the isomorphism

$$\mathrm{S}(\mathfrak{gl}_n)^{\mathrm{GL}_n} \cong \mathrm{Z}(\mathfrak{gl}_n),$$

where $\mathrm{Z}(\mathfrak{gl}_n)$ is the center of $\mathrm{U}(\mathfrak{gl}_n)$. Hence

$$\mathrm{Z}(\mathfrak{gl}_n) = \mathbb{C}\left[\varpi(\Delta_1), \ldots, \varpi(\Delta_n)\right].$$

This implies the isomorphism

$$S(\mathfrak{gl}_n)^{GL_n} \cong Z(\mathfrak{gl}_n),$$

where $Z(\mathfrak{gl}_n)$ is the center of $U(\mathfrak{gl}_n)$. Hence

$$Z(\mathfrak{gl}_n) = \mathbb{C}\left[\varpi(\Delta_1), \ldots, \varpi(\Delta_n)\right].$$

By Schur's Lemma, any element $z \in Z(\mathfrak{gl}_n)$ acts as scalar multiplication in any finite-dimensional simple $\mathfrak{gl}_n$-module.

This implies the isomorphism

$$\mathrm{S}(\mathfrak{gl}_n)^{\mathrm{GL}_n} \cong \mathrm{Z}(\mathfrak{gl}_n),$$

where $\mathrm{Z}(\mathfrak{gl}_n)$ is the center of $\mathrm{U}(\mathfrak{gl}_n)$. Hence

$$\mathrm{Z}(\mathfrak{gl}_n) = \mathbb{C}\left[\varpi(\Delta_1), \ldots, \varpi(\Delta_n)\right].$$

By Schur's Lemma, any element $z \in \mathrm{Z}(\mathfrak{gl}_n)$ acts as scalar multiplication in any finite-dimensional simple $\mathfrak{gl}_n$-module.

Question: What are the scalars corresponding to $\varpi(\Delta_i)$?

Any finite-dimensional simple $\mathfrak{gl}_n$-module L is generated

by a nonzero vector $\xi \in L$

Any finite-dimensional simple $\mathfrak{gl}_n$-module L is generated

by a nonzero vector $\xi \in L$ such that

$$E_{ij}\,\xi = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant n, \qquad \text{and}$$

$$E_{ii}\,\xi = \lambda_i\,\xi \qquad \text{for} \quad 1 \leqslant i \leqslant n,$$

Any finite-dimensional simple $\mathfrak{gl}_n$-module L is generated

by a nonzero vector $\xi \in L$ such that

$$E_{ij}\,\xi = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant n, \qquad \text{and}$$

$$E_{ii}\,\xi = \lambda_i\,\xi \qquad \text{for} \quad 1 \leqslant i \leqslant n,$$

for certain $\lambda_i \in \mathbb{C}$ satisfying the conditions $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+$.

Any finite-dimensional simple $\mathfrak{gl}_n$-module L is generated by a nonzero vector $\xi \in L$ such that

$$E_{ij}\,\xi = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant n, \qquad \text{and}$$

$$E_{ii}\,\xi = \lambda_i\,\xi \qquad \text{for} \quad 1 \leqslant i \leqslant n,$$

for certain $\lambda_i \in \mathbb{C}$ satisfying the conditions $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+$.

Any element $z \in \mathrm{Z}(\mathfrak{gl}_n)$ acts in L by multiplying each vector by a scalar $\chi(z)$.

Any finite-dimensional simple $\mathfrak{gl}_n$-module L is generated by a nonzero vector $\xi \in L$ such that

$$E_{ij}\,\xi = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant n, \qquad \text{and}$$

$$E_{ii}\,\xi = \lambda_i\,\xi \qquad \text{for} \quad 1 \leqslant i \leqslant n,$$

for certain $\lambda_i \in \mathbb{C}$ satisfying the conditions $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+$.

Any element $z \in \mathrm{Z}(\mathfrak{gl}_n)$ acts in L by multiplying each vector by a scalar $\chi(z)$. As a function of the parameters λ_i, the scalar $\chi(z)$ is a shifted symmetric polynomial in the variables $\lambda_1, \dots, \lambda_n$.

The polynomial $\chi(z)$ is symmetric in the shifted variables

$\lambda_1, \lambda_2 - 1, \dots, \lambda_n - n + 1$.

The polynomial $\chi(z)$ is symmetric in the shifted variables

$\lambda_1, \lambda_2 - 1, \dots, \lambda_n - n + 1$.

The map χ is the Harish-Chandra isomorphism between

$\mathrm{Z}(\mathfrak{gl}_n)$ and the algebra of shifted symmetric polynomials.

The polynomial $\chi(z)$ is symmetric in the shifted variables

$\lambda_1, \lambda_2 - 1, \dots, \lambda_n - n + 1$.

The map χ is the Harish-Chandra isomorphism between

$\mathrm{Z}(\mathfrak{gl}_n)$ and the algebra of shifted symmetric polynomials.

Algebraically independent generators:

elementary shifted symmetric polynomials

$$e_m^*(\lambda_1, \dots, \lambda_n) = \sum_{i_1 < \dots < i_m} \lambda_{i_1} (\lambda_{i_2} - 1) \dots (\lambda_{i_m} - m + 1)$$

with $m = 1, \dots, n$.

Stirling numbers

Stirling numbers

The Stirling number of the second kind $\left\{ {m \atop k} \right\}$ counts the number of partitions of the set $\{1, \ldots, m\}$ into k nonempty subsets.

Stirling numbers

The Stirling number of the second kind $\left\{ {m \atop k} \right\}$ counts the number of partitions of the set $\{1, \ldots, m\}$ into k nonempty subsets.

Equivalently, $k! \left\{ {m \atop k} \right\}$ is the number of surjective functions $\{1, \ldots, m\} \to \{1, \ldots, k\}$.

Stirling numbers

The Stirling number of the second kind $\left\{ {m \atop k} \right\}$ counts the number of partitions of the set $\{1, \ldots, m\}$ into k nonempty subsets.

Equivalently, $k! \left\{ {m \atop k} \right\}$ is the number of surjective functions $\{1, \ldots, m\} \to \{1, \ldots, k\}$.

Recurrence relation:

$$\left\{ {m \atop k} \right\} = \left\{ {m-1 \atop k-1} \right\} + k \left\{ {m-1 \atop k} \right\}.$$

Stirling triangle: $\left\{ {m \atop k} \right\}$ is in row m and column k

$$
\begin{array}{lllllll}
1 & & & & & & \\
1 & 1 & & & & & \\
1 & 3 & 1 & & & & \\
1 & 7 & 6 & 1 & & & \\
1 & 15 & 25 & 10 & 1 & & \\
1 & 31 & 90 & 65 & 15 & 1 & \\
1 & 63 & 301 & 350 & 140 & 21 & 1 \\
\vdots & & \vdots & & \vdots & & \vdots \quad \ddots
\end{array}
$$

Theorem. For the Harish-Chandra images we have

$$\chi : \varpi(\Delta_m) \mapsto \sum_{k=1}^{m} \left\{ {m \atop k} \right\} \binom{n}{m} \binom{n}{k}^{-1} e_k^*(\lambda_1, \ldots, \lambda_n).$$

Theorem. For the Harish-Chandra images we have

$$\chi : \varpi(\Delta_m) \mapsto \sum_{k=1}^{m} \left\{ {m \atop k} \right\} \binom{n}{m} \binom{n}{k}^{-1} e_k^*(\lambda_1, \dots, \lambda_n).$$

Proof. Regard the matrix $E = \big[E_{ij}\big]$ as the element

$$E = \sum_{i,j=1}^{n} e_{ij} \otimes E_{ij} \in \operatorname{End} \mathbb{C}^n \otimes \mathrm{U}\big(\mathfrak{gl}_n\big).$$

Theorem. For the Harish-Chandra images we have

$$\chi : \varpi(\Delta_m) \mapsto \sum_{k=1}^{m} \left\{ {m \atop k} \right\} \binom{n}{m} \binom{n}{k}^{-1} e_k^*(\lambda_1, \dots, \lambda_n).$$

Proof. Regard the matrix $E = \big[E_{ij}\big]$ as the element

$$E = \sum_{i,j=1}^{n} e_{ij} \otimes E_{ij} \in \operatorname{End} \mathbb{C}^n \otimes \mathrm{U}\big(\mathfrak{gl}_n\big).$$

Consider the algebra

$$\underbrace{\operatorname{End} \mathbb{C}^n \otimes \dots \otimes \operatorname{End} \mathbb{C}^n}_{m} \otimes \mathrm{U}\big(\mathfrak{gl}_n\big)$$

Theorem. For the Harish-Chandra images we have

$$\chi : \varpi(\Delta_m) \mapsto \sum_{k=1}^{m} \left\{ {m \atop k} \right\} \binom{n}{m} \binom{n}{k}^{-1} e_k^*(\lambda_1, \dots, \lambda_n).$$

Proof. Regard the matrix $E = \big[E_{ij}\big]$ as the element

$$E = \sum_{i,j=1}^{n} e_{ij} \otimes E_{ij} \in \operatorname{End} \mathbb{C}^n \otimes \mathrm{U}\big(\mathfrak{gl}_n\big).$$

Consider the algebra

$$\underbrace{\operatorname{End} \mathbb{C}^n \otimes \dots \otimes \operatorname{End} \mathbb{C}^n}_{m} \otimes \mathrm{U}\big(\mathfrak{gl}_n\big)$$

Observe that

$$\varpi(\Delta_m) = \operatorname{tr} A^{(m)} E_1 \dots E_m.$$

The defining relations of the algebra $\mathrm{U}(\mathfrak{gl}_n)$ can be written as

$$E_1 E_2 - E_2 E_1 = (E_1 - E_2) P$$

The defining relations of the algebra $\mathrm{U}(\mathfrak{gl}_n)$ can be written as

$$E_1 E_2 - E_2 E_1 = (E_1 - E_2) P$$

in the tensor product algebra

$$\mathrm{End}\,\mathbb{C}^n \otimes \mathrm{End}\,\mathbb{C}^n \otimes \mathrm{U}(\mathfrak{gl}_n).$$

The defining relations of the algebra $\mathrm{U}(\mathfrak{gl}_n)$ can be written as

$$E_1 E_2 - E_2 E_1 = (E_1 - E_2) P$$

in the tensor product algebra

$$\mathrm{End}\,\mathbb{C}^n \otimes \mathrm{End}\,\mathbb{C}^n \otimes \mathrm{U}(\mathfrak{gl}_n).$$

Introduce the extended algebra $\mathrm{U}(\mathfrak{gl}_n) \otimes \mathbb{C}[u, e^{\pm\partial_u}]$, where the element e^{∂_u} satisfies $e^{\partial_u} f(u) = f(u+1)\, e^{\partial_u}$.

The defining relations of the algebra $\mathrm{U}(\mathfrak{gl}_n)$ can be written as

$$E_1 E_2 - E_2 E_1 = (E_1 - E_2) P$$

in the tensor product algebra

$$\mathrm{End}\,\mathbb{C}^n \otimes \mathrm{End}\,\mathbb{C}^n \otimes \mathrm{U}(\mathfrak{gl}_n).$$

Introduce the extended algebra $\mathrm{U}(\mathfrak{gl}_n) \otimes \mathbb{C}[u, e^{\pm\partial_u}]$, where the element e^{∂_u} satisfies $e^{\partial_u} f(u) = f(u+1)\, e^{\partial_u}$.

Key observation:

$$M = (uI + E)\, e^{-\partial_u}$$

is a Manin matrix.

Hence

$$\operatorname{cdet} M = \operatorname{tr} A^{(n)} M_1 \ldots M_n.$$

Hence

$$\operatorname{cdet} M = \operatorname{tr} A^{(n)} M_1 \dots M_n.$$

This implies the relation for the Capelli determinant (1890),

$$\operatorname{cdet} \begin{bmatrix} u + E_{11} & E_{12} & \dots & E_{1n} \\ E_{21} & u + E_{22} - 1 & \dots & E_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ E_{n1} & \dots & \dots & u + E_{nn} - n + 1 \end{bmatrix}$$

$$= \operatorname{tr} A^{(n)} (u + E_1)(u + E_2 - 1) \dots (u + E_n - n + 1).$$

Hence

$$\operatorname{cdet} M = \operatorname{tr} A^{(n)} M_1 \dots M_n.$$

This implies the relation for the Capelli determinant (1890),

$$\operatorname{cdet} \begin{bmatrix} u+E_{11} & E_{12} & \dots & E_{1n} \\ E_{21} & u+E_{22}-1 & \dots & E_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ E_{n1} & \dots & \dots & u+E_{nn}-n+1 \end{bmatrix}$$

$$= \operatorname{tr} A^{(n)} (u+E_1)(u+E_2-1)\dots(u+E_n-n+1).$$

The Harish-Chandra image is $(u+\lambda_1)\dots(u+\lambda_n-n+1)$.

Similarly,

$$\chi : \operatorname{tr} A^{(m)} E_1 (E_2 - 1) \dots (E_m - m + 1) \mapsto e_m^*(\lambda_1, \dots, \lambda_n).$$

Similarly,

$$\chi : \operatorname{tr} A^{(m)} E_1 (E_2 - 1) \dots (E_m - m + 1) \mapsto e^*_m(\lambda_1, \dots, \lambda_n).$$

Using the identities for the Stirling numbers

$$x^m = \sum_{k=1}^{m} \left\{ {m \atop k} \right\} x(x-1)\dots(x-k+1),$$

Similarly,

$$\chi : \operatorname{tr} A^{(m)} E_1(E_2 - 1) \dots (E_m - m + 1) \mapsto e^*_m(\lambda_1, \dots, \lambda_n).$$

Using the identities for the Stirling numbers

$$x^m = \sum_{k=1}^{m} \left\{ {m \atop k} \right\} x(x-1) \dots (x-k+1),$$

we derive

$$\operatorname{tr} A^{(m)} E_1 \dots E_m = \operatorname{tr} A^{(m)} \sum_{k=1}^{m} \left\{ {m \atop k} \right\} E_1(E_2 - 1) \dots (E_k - k + 1).$$

Similarly,

$$\chi : \operatorname{tr} A^{(m)} E_1 (E_2 - 1) \dots (E_m - m + 1) \mapsto e^*_m(\lambda_1, \dots, \lambda_n).$$

Using the identities for the Stirling numbers

$$x^m = \sum_{k=1}^{m} \left\{ {m \atop k} \right\} x(x-1)\dots(x-k+1),$$

we derive

$$\operatorname{tr} A^{(m)} E_1 \dots E_m = \operatorname{tr} A^{(m)} \sum_{k=1}^{m} \left\{ {m \atop k} \right\} E_1(E_2 - 1)\dots(E_k - k + 1).$$

It remains to calculate the partial traces of $A^{(m)}$.

Feigin–Frenkel center

Feigin–Frenkel center

The affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ is the central extension

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$$

Feigin–Frenkel center

The affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ is the central extension

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$$

with the commutation relations for $X[r] = X\, t^r$:

$$\big[X[r], Y[s]\big] = [X, Y][r+s] + r\,\delta_{r,-s}\langle X, Y\rangle\, K.$$

Feigin–Frenkel center

The affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ is the central extension

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$$

with the commutation relations for $X[r] = X\,t^r$:

$$\big[X[r], Y[s]\big] = [X, Y][r+s] + r\,\delta_{r,-s}\langle X, Y\rangle\,K.$$

Note that $\tau = -\dfrac{d}{dt}$ is a derivation of $\widehat{\mathfrak{g}}$.

Consider the vacuum module at the critical level over $\widehat{\mathfrak{g}}$,

$$V(\mathfrak{g}) = \mathrm{U}(\widehat{\mathfrak{g}})/\mathrm{I},$$

where the left ideal I is generated by $\mathfrak{g}[t]$ and $K + h^{\vee}$.

Consider the vacuum module at the critical level over $\widehat{\mathfrak{g}}$,

$$V(\mathfrak{g}) = \mathrm{U}(\widehat{\mathfrak{g}})/\mathrm{I},$$

where the left ideal I is generated by $\mathfrak{g}[t]$ and $K + h^{\vee}$.

The Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ is defined by

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \{v \in V(\mathfrak{g}) \mid \mathfrak{g}[t]\, v = 0\}.$$

Consider the vacuum module at the critical level over $\widehat{\mathfrak{g}}$,

$$V(\mathfrak{g}) = \mathrm{U}(\widehat{\mathfrak{g}})/\mathrm{I},$$

where the left ideal I is generated by $\mathfrak{g}[t]$ and $K + h^\vee$.

The Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ is defined by

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \{v \in V(\mathfrak{g}) \mid \mathfrak{g}[t]\, v = 0\}.$$

We have $V(\mathfrak{g}) \cong \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ as vector spaces,

Consider the vacuum module at the critical level over $\widehat{\mathfrak{g}}$,

$$V(\mathfrak{g}) = \mathrm{U}(\widehat{\mathfrak{g}})/\mathrm{I},$$

where the left ideal I is generated by $\mathfrak{g}[t]$ and $K + h^\vee$.

The Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ is defined by

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \{v \in V(\mathfrak{g}) \mid \mathfrak{g}[t]\, v = 0\}.$$

We have $V(\mathfrak{g}) \cong \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ as vector spaces, and

$\mathfrak{z}(\widehat{\mathfrak{g}})$ is a τ-invariant commutative subalgebra of $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$.

Equivalently, $\mathfrak{z}(\widehat{\mathfrak{g}})$ can be defined as the centralizer of the canonical Segal–Sugawara vector

$$S = \sum_{i=1}^{d} X_i[-1]^2,$$

where $X_1, \dots, X_d$ is an orthonormal basis of $\mathfrak{g}$.

Equivalently, $\mathfrak{z}(\widehat{\mathfrak{g}})$ can be defined as the centralizer of

the canonical Segal–Sugawara vector

$$S = \sum_{i=1}^{d} X_i[-1]^2,$$

where $X_1, \dots, X_d$ is an orthonormal basis of $\mathfrak{g}$.

As a differential algebra, it possesses free generators

$S_1, \dots, S_n$ (a complete set of Segal–Sugawara vectors),

where $n = \operatorname{rank} \mathfrak{g}$ and $S_1 = S$.

Equivalently, $\mathfrak{z}(\widehat{\mathfrak{g}})$ can be defined as the centralizer of the canonical Segal–Sugawara vector

$$S = \sum_{i=1}^{d} X_i[-1]^2,$$

where $X_1, \dots, X_d$ is an orthonormal basis of $\mathfrak{g}$.

As a differential algebra, it possesses free generators $S_1, \dots, S_n$ (a complete set of Segal–Sugawara vectors), where $n = \operatorname{rank} \mathfrak{g}$ and $S_1 = S$.

[B. Feigin and E. Frenkel 1992, L. Rybnikov 2008].

- The Segal–Sugawara vectors $S_1, \dots, S_n$ give rise to higher order Hamiltonians in the Gaudin model.

- The Segal–Sugawara vectors $S_1, \dots, S_n$ give rise to higher order Hamiltonians in the Gaudin model.
- The eigenvalues of the Hamiltonians on the Bethe vectors are found from the Harish-Chandra images of $S_1, \dots, S_n$.

- The Segal–Sugawara vectors $S_1, \dots, S_n$ give rise to higher order Hamiltonians in the Gaudin model.

- The eigenvalues of the Hamiltonians on the Bethe vectors are found from the Harish-Chandra images of $S_1, \dots, S_n$.

- Applying homomorphisms $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big) \to \mathrm{U}(\mathfrak{g})$ one gets commutative subalgebras of $\mathrm{U}(\mathfrak{g})$ thus solving Vinberg's quantization problem.

Type A

Type A

Working with the algebra

$$\underbrace{\operatorname{End}\mathbb{C}^n \otimes \ldots \otimes \operatorname{End}\mathbb{C}^n}_{m} \otimes \mathrm{U}\big(t^{-1}\mathfrak{gl}_n[t^{-1}]\big),$$

Type A

Working with the algebra

$$\underbrace{\operatorname{End}\mathbb{C}^n \otimes \ldots \otimes \operatorname{End}\mathbb{C}^n}_{m} \otimes \mathrm{U}\big(t^{-1}\mathfrak{gl}_n[t^{-1}]\big),$$

write

$$S_m = \operatorname{tr} A^{(m)}\big(\tau + E_1[-1]\big)\ldots\big(\tau + E_m[-1]\big)\,1,$$

Type A

Working with the algebra

$$\underbrace{\operatorname{End}\mathbb{C}^n \otimes \ldots \otimes \operatorname{End}\mathbb{C}^n}_{m} \otimes \mathrm{U}\big(t^{-1}\mathfrak{gl}_n[t^{-1}]\big),$$

write

$$S_m = \operatorname{tr} A^{(m)}\big(\tau + E_1[-1]\big) \ldots \big(\tau + E_m[-1]\big)\, 1,$$

where

$$E[r] = \sum_{i,j=1}^{n} e_{ij} \otimes E_{ij}[r] \in \operatorname{End}\mathbb{C}^n \otimes \mathrm{U}\big(t^{-1}\mathfrak{gl}_n[t^{-1}]\big).$$

Key lemma: $M = \tau I + E[-1]$ is a Manin matrix.

Key lemma: $M = \tau I + E[-1]$ is a Manin matrix.

Hence the elements S_m are also found by

$$\operatorname{cdet} \begin{bmatrix} \tau + E_{11}[-1] & \dots & E_{1n}[-1] \\ \vdots & \ddots & \vdots \\ E_{n1}[-1] & \dots & \tau + E_{nn}[-1] \end{bmatrix} = \tau^{n} + S_1 \tau^{n-1} + \dots + S_n.$$

Key lemma: $M = \tau I + E[-1]$ is a Manin matrix.

Hence the elements S_m are also found by

$$\operatorname{cdet} \begin{bmatrix} \tau + E_{11}[-1] & \dots & E_{1n}[-1] \\ \vdots & \ddots & \vdots \\ E_{n1}[-1] & \dots & \tau + E_{nn}[-1] \end{bmatrix} = \tau^{n} + S_1 \tau^{n-1} + \dots + S_n.$$

Theorem $S_1, \dots, S_n$ are free generators of $\mathfrak{z}(\widehat{\mathfrak{gl}}_n)$.

[A. Chervov and D. Talalaev 2006, A. Chervov and A. M. 2009].

Eliminate $\tau = -\dfrac{d}{dt}$ to get

$$S_m = \operatorname{tr} A^{(m)} \big(\tau + E_1[-1]\big) \dots \big(\tau + E_m[-1]\big)\, 1$$

$$= \sum_{\lambda \vdash m} c_\lambda \operatorname{tr} A^{(m)} E_1[-\lambda_1] \dots E_\ell[-\lambda_\ell],$$

Eliminate $\tau = -\dfrac{d}{dt}$ to get

$$S_m = \operatorname{tr} A^{(m)} \big(\tau + E_1[-1]\big) \dots \big(\tau + E_m[-1]\big)\, 1$$

$$= \sum_{\lambda \vdash m} c_\lambda \operatorname{tr} A^{(m)} E_1[-\lambda_1] \dots E_\ell[-\lambda_\ell],$$

where the parts of partitions λ are $\lambda_1 \geqslant \dots \geqslant \lambda_\ell > 0$ and

Eliminate $\tau = -\dfrac{d}{dt}$ to get

$$S_m = \operatorname{tr} A^{(m)} \big(\tau + E_1[-1]\big) \dots \big(\tau + E_m[-1]\big)\, 1$$

$$= \sum_{\lambda \vdash m} c_\lambda \operatorname{tr} A^{(m)} E_1[-\lambda_1] \dots E_\ell[-\lambda_\ell],$$

where the parts of partitions λ are $\lambda_1 \geqslant \dots \geqslant \lambda_\ell > 0$ and

c_λ is the number of permutations of $\{1, \dots, m\}$ of cycle type λ.

Theorem [O. Yakimova 2019, A. M. 2020].

The Segal–Sugawara vectors are given by

$$S_m = \sum_{\lambda \vdash m} \binom{N}{m} \binom{N}{\ell}^{-1} c_\lambda \operatorname{tr} A^{(\ell)} E_1[-\lambda_1] \dots E_\ell[-\lambda_\ell].$$

Theorem [O. Yakimova 2019, A. M. 2020].

The Segal–Sugawara vectors are given by

$$S_m = \sum_{\lambda \vdash m} \binom{N}{m} \binom{N}{\ell}^{-1} c_\lambda \operatorname{tr} A^{(\ell)} E_1[-\lambda_1] \dots E_\ell[-\lambda_\ell].$$

Applications [L. Rybnikov 2006]: for any $z \in \mathbb{C}^\times$ and $\mu \in \mathfrak{gl}_n^*$

the image of $\mathfrak{z}(\widehat{\mathfrak{gl}}_n)$ under the homomorphism

$$\varrho_{\mu,z} : \mathrm{U}\big(t^{-1}\mathfrak{gl}_n[t^{-1}]\big) \to \mathrm{U}(\mathfrak{gl}_n), \qquad X[r] \mapsto X z^r + \delta_{r,-1}\, \mu(X),$$

is a commutative subalgebra $\mathcal{A}_\mu$ of $\mathrm{U}(\mathfrak{gl}_n)$ independent of z.