Manin matrices

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Plan of lectures

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- Origins and motivations.

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- Basic properties of Manin matrices.

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- Examples and applications to Casimir elements.

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- Examples and applications to Casimir elements.
- Generalizations: q-Manin and super-Manin matrices.

References

References

A. Chervov, G. Falqui and V. Rubtsov, Algebraic properties of Manin matrices 1, Adv. Appl. Math. **43** (2009), 239–315.

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A. Chervov, G. Falqui and V. Rubtsov, Algebraic properties of Manin matrices 1, Adv. Appl. Math. **43** (2009), 239–315.

A. Molev, Sugawara operators for classical Lie algebras, AMS, 2018; Chapter 3.

Quantum groups

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By the seminal works of V. Drinfeld (1985) and M. Jimbo (1985), the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$ admits a deformation $\mathrm{U}_q(\mathfrak{g})$ in the class of Hopf algebras.

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A detailed review of the theory and applications:

V. Chari and A. Pressley, *A guide to quantum groups*, 1994.

Basic example

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[L. Faddeev and L. Takhtajan 1986].

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A 2×2 matrix is q-Manin if the elements x' and y' defined by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

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satisfy $y'x' = qx'y'$. The defining relations for $\mathrm{Fun}_q(\mathrm{Mat}_2)$ are equivalent to the conditions that the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and its transpose are q-Manin matrices.

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$$[ax + by, cx + dy] = [a, c]\, x^2 + \big([a, d] + [b, c]\big)\, xy + [b, d]\, y^2.$$

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This leads to the definition of Manin matrices:

$$[a, c] = [b, d] = 0 \qquad \text{and} \qquad [a, d] = [c, b].$$

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Look for $n \times n$ matrices $M = [M_{ij}]$ over an associative algebra $\mathcal{A}$,

such that $x_1', \ldots, x_n'$ defined by

$$\begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} M_{11} & \ldots & M_{1n} \\ \vdots & \vdots & \vdots \\ M_{n1} & \ldots & M_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

commute.

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[Open question in the super case.]

Determinants

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Introduce the column-determinant of a matrix M by

$$\operatorname{cdet} M = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn} \sigma \cdot M_{\sigma(1)\,1} \dots M_{\sigma(n)\,n}.$$

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Exercise. Suppose that M is a Manin matrix. Verify that $\operatorname{cdet} M$ possesses usual properties of determinant: it changes sign if two rows or two columns are swapped.

Tensor techniques

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For any associative algebra $\mathcal{A}$ we have an algebra isomorphism

$$\mathrm{Mat}_n(\mathcal{A}) \cong \operatorname{End}\mathbb{C}^n \otimes \mathcal{A}.$$

We may regard the matrix M over $\mathcal{A}$ as the element

$$M = \sum_{i,j=1}^{n} e_{ij} \otimes M_{ij} \in \operatorname{End}\mathbb{C}^n \otimes \mathcal{A}.$$

Consider the algebra

$$\underbrace{\operatorname{End}\mathbb{C}^n \otimes \ldots \otimes \operatorname{End}\mathbb{C}^n}_{k} \otimes \mathcal{A}$$

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and for $a = 1, \ldots, k$ set

$$M_a = \sum_{i,j=1}^{n} \underbrace{1 \otimes \ldots \otimes 1}_{a-1} \otimes e_{ij} \otimes \underbrace{1 \otimes \ldots \otimes 1}_{k-a} \otimes M_{ij},$$

where 1 is the identity matrix.

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In particular, we have the permutation operator

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Exercise. Verify that P is given by

$$P = \sum_{i,j=1}^{n} e_{ij} \otimes e_{ji} \in \operatorname{End}\mathbb{C}^n \otimes \operatorname{End}\mathbb{C}^n.$$

In general, for the transposition $(a\,b) \in \mathfrak{S}_k$ we have $(a\,b) \mapsto P_{ab}$,

where

$$P_{ab} = \sum_{i,j=1}^{n} \underbrace{1 \otimes \ldots \otimes 1}_{a-1} \otimes e_{ij} \otimes \underbrace{1 \otimes \ldots \otimes 1}_{b-a-1} \otimes e_{ji} \otimes \underbrace{1 \otimes \ldots \otimes 1}_{k-b}.$$

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Elements of the group algebra $\mathbb{C}[\mathfrak{S}_k]$ are then represented as operators in $(\mathbb{C}^n)^{\otimes k}$; that is, as elements of the algebra

$$\mathrm{End}\left((\mathbb{C}^n)^{\otimes k}\right) \cong \underbrace{\mathrm{End}\,\mathbb{C}^n \otimes \ldots \otimes \mathrm{End}\,\mathbb{C}^n}_{k}.$$

Exercise. Verify the relations in the algebra $\mathrm{End}\,(\mathbb{C}^n)^{\otimes k} \otimes \mathcal{A}$:

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More generally, for any $\sigma \in \mathfrak{S}_k$ let P_σ denote its image under the action on the tensor product space $(\mathbb{C}^n)^{\otimes k}$.

Show that

$$P_\sigma\, M_a = M_{\sigma(a)}\, P_\sigma.$$

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Lemma. Each of following relations provides an equivalent definition of Manin matrices:

$$(1 - P)\, M_1 M_2\, (1 + P) = 0,$$

$$(1 - P)\, (M_1 M_2 - M_2 M_1) = 0,$$

$$(M_1 M_2 - M_2 M_1)\, (1 + P) = 0.$$

Proof. The relations are equivalent to each other by the

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$$M_1 M_2 = \sum_{i,j,k,l=1}^{n} e_{ij} \otimes e_{kl} \otimes M_{ij} M_{kl}.$$

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$$M_1 M_2 = \sum_{i,j,k,l=1}^{n} e_{ij} \otimes e_{kl} \otimes M_{ij} M_{kl}.$$

Hence, using the formula for P we get

$$P M_1 M_2 = \sum_{i,j,k,l=1}^{n} e_{kj} \otimes e_{il} \otimes M_{ij} M_{kl},$$

$$M_1 M_2 P = \sum_{i,j,k,l=1}^{n} e_{il} \otimes e_{kj} \otimes M_{ij} M_{kl},$$

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and

$$P M_1 M_2 P = \sum_{i,j,k,l=1}^{n} e_{kl} \otimes e_{ij} \otimes M_{ij} M_{kl}.$$

Therefore, taking the coefficient of the basis vector $e_{ij} \otimes e_{kl}$ on the left hand side of

$$(1 - P) M_1 M_2 (1 + P)$$

$$= M_1 M_2 - P M_1 M_2 + M_1 M_2 P - P M_1 M_2 P$$

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we find that the first relation is equivalent to the defining relations for Manin matrices.

Remark on a new Hecke-type algebra

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together with

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Open problem: understand this “Hecke–Manin” algebra.

Denote by $H^{(k)}$ and $A^{(k)}$ the respective images of the

symmetrizer and anti-symmetrizer

$$\frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}\sigma \qquad \text{and} \qquad \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}\operatorname{sgn}\sigma\cdot\sigma.$$

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Example.

$$H^{(2)}=\frac{1}{2}\,(1+P),\qquad A^{(2)}=\frac{1}{2}\,(1-P).$$

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$$A^{(k)} = \frac{1}{k} A^{(k-1)} - \frac{k-1}{k} A^{(k-1)} P_{k-1\,k} A^{(k-1)}$$

and

$$H^{(k)} = \frac{1}{k} H^{(k-1)} + \frac{k-1}{k} H^{(k-1)} P_{k-1\,k} H^{(k-1)}.$$

Proof. We have (verify!)

$$A^{(k)} = \frac{1}{k} A^{(k-1)} (1 - P_{1\,k} - \cdots - P_{k-1\,k}).$$

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Multiply both sides by $A^{(k-1)}$ from the right and use the relations

$$A^{(k)} A^{(k-1)} = A^{(k)}$$

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Multiply both sides by $A^{(k-1)}$ from the right and use the relations

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and

$$A^{(k-1)} P_{a\,k} A^{(k-1)} = A^{(k-1)} P_{k-1\,k} A^{(k-1)}$$

for $1 \leqslant a < k$.

Proposition.

If M is a Manin matrix, then we have the identities in the algebra $\mathrm{End}\,(\mathbb{C}^n)^{\otimes k} \otimes \mathcal{A}$:

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and

$$H^{(k)} M_1 \dots M_k H^{(k)} = M_1 \dots M_k H^{(k)}.$$

Moreover,

$$A^{(n)} M_1 \dots M_n = A^{(n)}\, \mathrm{cdet}\, M.$$

Proof. To prove the first relation it suffices to show that for any element $\sigma \in \mathfrak{S}_k$ we have

$$A^{(k)} M_1 \dots M_k P_\sigma = \operatorname{sgn} \sigma \cdot A^{(k)} M_1 \dots M_k,$$

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where P_σ is the image of $\sigma \in \mathfrak{S}_k$.

Since the group $\mathfrak{S}_k$ is generated by the adjacent transpositions, it is enough to verify the relation for the elements $\sigma = (a\; a+1)$ with $a = 1, \dots, k-1$.

Hence we only need to consider the case $k = 2$. However, the relation with $\sigma = (1\,2)$ reads

$$\frac{1-P}{2} M_1 M_2 P = -\frac{1-P}{2} M_1 M_2$$

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The proof of the second relation reduces to checking that for any $\sigma \in \mathfrak{S}_k$

$$P_\sigma M_1 \dots M_k H^{(k)} = M_1 \dots M_k H^{(k)}.$$

This follows again from the defining relations written in the form

$$P\,M_1 M_2 \frac{1+P}{2} = M_1 M_2 \frac{1+P}{2}.$$

By the trace we will mean the linear map

$$\operatorname{tr} : \operatorname{End} \mathbb{C}^n \to \mathbb{C}, \qquad e_{ij} \mapsto \delta_{ij}.$$

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Furthermore, for any $a \in \{1, \dots, k\}$ the partial trace tr_a will be understood as the linear map

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The full trace $\operatorname{tr} = \operatorname{tr}_{1,\ldots,k}$ is the composition $\operatorname{tr}_1 \circ \cdots \circ \operatorname{tr}_k$.

Exercises. Show that

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Lemma. Suppose that two elements

$$X = \sum e_{i_1 j_1} \otimes \ldots \otimes e_{i_k j_k} \otimes X^{i_1 \ldots i_k}_{j_1 \ldots j_k} \qquad \text{and}$$

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$$X^{i_1 \ldots i_k}_{j_1 \ldots j_k} \, Y^{m_1 \ldots m_k}_{l_1 \ldots l_k} = Y^{m_1 \ldots m_k}_{l_1 \ldots l_k} \, X^{i_1 \ldots i_k}_{j_1 \ldots j_k}$$

for all values of the indices.

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$$X^{\,i_1\ldots i_k}_{\,j_1\ldots j_k}\,Y^{\,m_1\ldots m_k}_{\,l_1\ldots l_k}=Y^{\,m_1\ldots m_k}_{\,l_1\ldots l_k}\,X^{\,i_1\ldots i_k}_{\,j_1\ldots j_k}$$

for all values of the indices. Then

$$\mathrm{tr}\,XY=\mathrm{tr}\,YX.$$

MacMahon Master Theorem

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For any $n \times n$ matrix M over an associative algebra $\mathcal{A}$ set

$$\mathrm{Ferm} = 1 + \sum_{k=1}^{n} (-1)^k \operatorname{tr} A^{(k)} M_1 \ldots M_k,$$

$$\mathrm{Bos} = 1 + \sum_{k=1}^{\infty} \operatorname{tr} H^{(k)} M_1 \ldots M_k.$$

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Theorem [Garoufalidis–Lê–Zeilberger 2006].

If M is a Manin matrix, then

$$\mathrm{Bos} \times \mathrm{Ferm} = 1.$$

Proof.

It is sufficient to show that for any integer $1 \leqslant k \leqslant N$ we have the identity in the algebra $\operatorname{End}(\mathbb{C}^n)^{\otimes k} \otimes \mathcal{A}$

$$\sum_{r=0}^{k} (-1)^{k-r} \operatorname{tr}_{1,\dots,r} H^{(r)} M_1 \dots M_r \times \operatorname{tr}_{r+1,\dots,k} A^{\{r+1,\dots,k\}} M_{r+1} \dots M_k = 0,$$

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where $A^{\{r+1,\dots,k\}}$ denotes the anti-symmetrizer over the copies of $\operatorname{End}\mathbb{C}^n$ labeled by $r+1,\dots,k$ (with the identity components in the first r copies).

The identity can be written as

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We will show that the left hand side of (1) remains unchanged after the replacement of the product of the symmetrizer and anti-symmetrizer $H^{(r)}A^{\{r+1,\dots,k\}}$ by

$$\frac{r(k-r+1)}{k} H^{(r)} A^{\{r,\dots,k\}} + \frac{(r+1)(k-r)}{k} H^{(r+1)} A^{\{r+1,\dots,k\}}.$$

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If this is true, then (1) vanishes after the replacement since we get a telescoping sum equal to zero.

Working with $H^{(r+1)}A^{\{r+1,\ldots,k\}}$, use the recurrence relation

$$H^{(r+1)} = \frac{1}{r+1}\,H^{(r)} + \frac{r}{r+1}\,H^{(r)}P_{r\,r+1}H^{(r)}.$$

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By the cyclic property of the trace, we get

$$\operatorname{tr} H^{(r)}P_{r\,r+1}H^{(r)}A^{\{r+1,\dots,k\}}M_1\dots M_k$$

$$= \operatorname{tr} P_{r\,r+1}A^{\{r+1,\dots,k\}}H^{(r)}M_1\dots M_kH^{(r)}.$$

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Hence, by the second identity in the Proposition, this equals

$$\operatorname{tr} P_{r\,r+1}A^{\{r+1,\dots,k\}}M_1\dots M_kH^{(r)}$$

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we have

$$M_{ij}M_{kl} - M_{kj}M_{il} = M_{kl}M_{ij} - M_{il}M_{kj}.$$

Equivalently, M is a Manin matrix, if and only if in the product algebra

$$\mathrm{End}\,\mathbb{C}^n \otimes \mathrm{End}\,\mathbb{C}^n \otimes \mathcal{A}$$

we have

$$(1 - P)\,(M_1 M_2 - M_2 M_1) = 0,$$

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$$(1 - P)\,(M_1 M_2 - M_2 M_1) = 0,$$

where

$$M_1 = \sum_{i,j=1}^{n} e_{ij} \otimes 1 \otimes M_{ij}$$

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Noncommutative characteristic polynomial

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Proposition. If M is a Manin matrix, then

$$\operatorname{cdet}(1 + tM) = \sum_{k=0}^{n} t^k \operatorname{tr} A^{(k)} M_1 \dots M_k,$$

$$\big[\operatorname{cdet}(1 - tM)\big]^{-1} = \sum_{k=0}^{\infty} t^k \operatorname{tr} H^{(k)} M_1 \dots M_k.$$

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Proof. Write

$$A^{(k)} M_1\dots M_k=\sum_{I,J} e_{i_1 j_1}\otimes\dots\otimes e_{i_k j_k}\otimes M^{\,i_1\dots\, i_k}_{\,j_1\dots j_k},$$

summed over all k-tuples of indices $I=(i_1,\dots,i_k)$ and

$J=(j_1,\dots,j_k)$ from $\{1,\dots,n\}$, where $M^{\,i_1\dots\, i_k}_{\,j_1\dots j_k}\in\mathcal{A}$.

For each $a = 1, \dots, k-1$ we have

$$P_{a\,a+1} A^{(k)} M_1 \dots M_k = -A^{(k)} M_1 \dots M_k = A^{(k)} M_1 \dots M_k P_{a\,a+1}.$$

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$$\operatorname{tr} A^{(k)} M_1 \ldots M_k = \sum_I M^{\,i_1 \ldots i_k}_{\,i_1 \ldots i_k} = k! \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} M^{\,i_1 \ldots i_k}_{\,i_1 \ldots i_k}$$

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which coincides with the coefficient of t^k in $\operatorname{cdet}(1 + tM)$.

Cayley–Hamilton identity

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Define the comatrix for a Manin matrix M as the matrix $\widehat{M}$ with the entries in the algebra $\mathcal{A}$ defined by

$$\widehat{M}_{ij} = (-1)^{i+j} \operatorname{cdet} M^{ji},$$

where M^{ji} is the matrix obtained from M by deleting row j and column i.

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where M^{ji} is the matrix obtained from M by deleting row j and column i.

Lemma. We have the relation

$$\widehat{M} M = \big(\operatorname{cdet} M\big)\, 1.$$

Proof. First observe that the definition of the comatrix can be written equivalently in the matrix form as

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Indeed,

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so that the matrix relation is equivalent to the equality of the matrix coefficients corresponding to the basis vectors of the form

$$e_1 \otimes \dots \otimes \widehat{e}_i \otimes \dots \otimes e_n \otimes e_j, \qquad i, j \in \{1, \dots, n\}.$$

Apply both sides of the matrix relation to such a vector and compare the coefficients of the vector

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as required. Now, by the Proposition,

$$A^{(n)}\operatorname{cdet} M = A^{(n)} M_1\ldots M_n = A^{(n)}\widehat{M}_n M_n.$$

On applying both sides to the above vectors we get the Lemma.

Theorem.

For a Manin matrix M set

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$$(\widehat{u\,1 - M})\,(u - M) = C(u)\,1.$$

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[Open problem in the super case.]

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Proof. Since

$$A^{(n)} M_n \dots M_1 = A^{(n)} \mathrm{cdet}\, M,$$

we have (assuming $n \geqslant 2$)

$$(\mathrm{cdet}\, M)^{-1} A^{(n)} M_n \dots M_3 = A^{(n)} M_1^{-1} M_2^{-1}$$

so that the right hand side is unchanged after the multiplication by $-P_{12}$ from the right.

Hence,

$$A^{(n)}(M_1^{-1}M_2^{-1} - M_2^{-1}M_1^{-1}) = 0.$$

Hence,

$$A^{(n)}(M_1^{-1}M_2^{-1} - M_2^{-1}M_1^{-1}) = 0.$$

Taking the partial trace $\mathrm{tr}_{3,\dots,n}$ we get

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[No proof is known in the super case.]

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Theorem. If M is a Manin matrix, then

$$\frac{d}{dt}\,\mathrm{cdet}(1+tM)=\mathrm{cdet}(1+tM)\sum_{k=0}^{\infty}(-t)^k\,\mathrm{tr}\,M^{k+1}.$$

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$$\frac{d}{dt}\,\mathrm{cdet}(1+tM) = \mathrm{cdet}(1+tM)\sum_{k=0}^{\infty}(-t)^k\,\mathrm{tr}\,M^{k+1}.$$

Proof. Since $1+tM$ is also a Manin matrix, we have

$$A^{(n)}(1+tM_1)\dots(1+tM_n) = A^{(n)}\mathrm{cdet}\,(1+tM).$$

Calculate the derivative of both sides over t:

$$\sum_{a=1}^{n} A^{(n)}(1+tM_1)\dots M_a\dots(1+tM_n) = A^{(n)}\frac{d}{dt}\,\mathrm{cdet}(1+tM).$$

Replace the factor M_a by $t^{-1}(1 + tM_a) - t^{-1}$, then take the trace of both sides over all n copies of $\operatorname{End}\mathbb{C}^n$ to get

$$n t^{-1} \operatorname{cdet}(1+tM) - t^{-1} \sum_{a=1}^{n} \operatorname{tr} A^{(n)} (1+tM_1) \dots (\widehat{1 + tM_a}) \dots (1+tM_n) = \frac{d}{dt} \operatorname{cdet}(1 + tM).$$

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$$= \frac{d}{dt}\,\mathrm{cdet}(1 + tM).$$

Observe that for each value of a the corresponding term in the sum coincides with the term for $a = n$ which equals

$$\mathrm{tr}\, A^{(n)} (1 + tM_1) \dots (1 + tM_{n-1}).$$

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It can be written in the form

$$\operatorname{cdet}(1+tM)\sum_{k=0}^{\infty}(-t)^{k}\operatorname{tr}M^{k+1}=\frac{d}{dt}\operatorname{cdet}(1+tM),$$

as required.

Applications: Casimir elements

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The group GL_n acts on $\mathfrak{gl}_n$ by conjugation: $X \mapsto g X g^{-1}$,

and the action extends to the symmetric algebra $\mathrm{S}(\mathfrak{gl}_n)$ which

can be viewed as the algebra of polynomials in n^2 variables E_{ij}.

Consider the matrix

$$E = \begin{bmatrix} E_{11} & \dots & E_{1n} \\ \vdots & \vdots & \vdots \\ E_{n1} & \dots & E_{nn} \end{bmatrix}$$

with entries in the symmetric algebra $\mathrm{S}(\mathfrak{gl}_n)$.

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We have

$$\mathrm{S}(\mathfrak{gl}_n)^{\mathrm{GL}_n} = \mathbb{C}[\Delta_1, \dots, \Delta_n].$$

The universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$ is the associative algebra with n^2 generators E_{ij} and the defining relations

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The symmetrization map

$$\varpi : \mathrm{S}(\mathfrak{gl}_n) \xrightarrow{\sim} \mathrm{U}(\mathfrak{gl}_n),$$

is a GL_n-module isomorphism, defined by

$$\varpi : X_1 \dots X_k \mapsto \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} X_{\sigma(1)} \dots X_{\sigma(k)}, \qquad X_i \in \mathfrak{gl}_n,$$

[Poincaré–Birkhoff–Witt Theorem].

This implies the isomorphism

$$\mathrm{S}(\mathfrak{gl}_n)^{\mathrm{GL}_n} \cong \mathrm{Z}(\mathfrak{gl}_n),$$

where $\mathrm{Z}(\mathfrak{gl}_n)$ is the center of $\mathrm{U}(\mathfrak{gl}_n)$.

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Question: What are the scalars corresponding to $\varpi(\Delta_i)$?

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Any finite-dimensional simple $\mathfrak{gl}_n$-module L is generated by a nonzero vector $\xi \in L$ such that

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for certain $\lambda_i \in \mathbb{C}$ satisfying the conditions $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+$.

Any element $z \in \mathrm{Z}(\mathfrak{gl}_n)$ acts in L by multiplying each vector by a scalar $\chi(z)$. As a function of the parameters λ_i, the scalar $\chi(z)$ is a shifted symmetric polynomial in the variables $\lambda_1, \dots, \lambda_n$.

The polynomial $\chi(z)$ is symmetric in the shifted variables

$\lambda_1, \lambda_2 - 1, \ldots, \lambda_n - n + 1$.

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Algebraically independent generators:

elementary shifted symmetric polynomials

$$e^*_m(\lambda_1, \dots, \lambda_n) = \sum_{i_1 < \dots < i_m} \lambda_{i_1} (\lambda_{i_2} - 1) \dots (\lambda_{i_m} - m + 1)$$

with $m = 1, \dots, n$.

The Stirling number of the second kind $\left\{ m \atop k \right\}$ counts the number of partitions of the set $\{1, \ldots, m\}$ into k nonempty subsets.

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Theorem. For the Harish-Chandra images we have

$$\chi : \varpi(\Delta_m) \mapsto \sum_{k=1}^{m} \left\{ m \atop k \right\} \binom{n}{m} \binom{n}{k}^{-1} e_k^*(\lambda_1, \dots, \lambda_n).$$

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Proof. Regard the matrix $E = \big[E_{ij}\big]$ as the element

$$E = \sum_{i,j=1}^{n} e_{ij} \otimes E_{ij} \in \operatorname{End} \mathbb{C}^n \otimes \mathrm{U}\big(\mathfrak{gl}_n\big).$$

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Observe that

$$\varpi(\Delta_m) = \operatorname{tr} A^{(m)} E_1 \ldots E_m.$$

The defining relations of the algebra $\mathrm{U}(\mathfrak{gl}_n)$ can be written as

$$E_1 E_2 - E_2 E_1 = (E_1 - E_2) P$$

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Introduce the extended algebra $\mathrm{U}(\mathfrak{gl}_n) \otimes \mathbb{C}[u, e^{\pm\partial_u}]$, where the element e^{∂_u} satisfies $e^{\partial_u} f(u) = f(u+1)\, e^{\partial_u}$.

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Key observation:

$$M = (u\,1 + E)\, e^{-\partial_u}$$

is a Manin matrix.

Hence

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This implies the relation for the Capelli determinant (1890),

$$\operatorname{cdet} \begin{bmatrix} u + E_{11} & E_{12} & \dots & E_{1n} \\ E_{21} & u + E_{22} - 1 & \dots & E_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ E_{n1} & \dots & \dots & u + E_{nn} - n + 1 \end{bmatrix}$$

$$= \operatorname{tr} A^{(n)} (u + E_1)(u + E_2 - 1) \dots (u + E_n - n + 1).$$

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The Harish-Chandra image is $(u + \lambda_1) \dots (u + \lambda_n - n + 1)$.

Similarly,

$$\chi : \operatorname{tr} A^{(m)} E_1 (E_2 - 1) \ldots (E_m - m + 1) \mapsto e^*_m(\lambda_1, \ldots, \lambda_n).$$

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$$x^m = \sum_{k=1}^{m} \left\{ {m \atop k} \right\} x(x-1)\ldots(x-k+1),$$

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It remains to calculate the partial traces of $A^{(m)}$.

Further examples

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Consider the algebra $\mathcal{A} = \mathrm{U}\big(t^{-1}\mathfrak{gl}_n[t^{-1}]\big)$ and let $\tau = -\dfrac{d}{dt}$.

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Lemma. The matrix $M = \tau\,1 + E[-1]$ is a Manin matrix.

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Consider the algebra $\mathcal{A} = \mathrm{U}\big(t^{-1}\mathfrak{gl}_n[t^{-1}]\big)$ and let $\tau = -\dfrac{d}{dt}$.

Lemma. The matrix $M = \tau\, 1 + E[-1]$ is a Manin matrix.

This fact is essential in the constructions of Sugawara operators for $\mathfrak{gl}_n$.

The Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ for $\mathfrak{gl}_n$ is an associative algebra with countably many generators $t_{ij}^{(1)},\ t_{ij}^{(2)},\dots$ where $i,j=1,\dots,n$, and the defining relations

$$[t_{ij}^{(r+1)},t_{kl}^{(s)}]-[t_{ij}^{(r)},t_{kl}^{(s+1)}]=t_{kj}^{(r)}t_{il}^{(s)}-t_{kj}^{(s)}t_{il}^{(r)},$$

where $r,s=0,1,\dots$ and $t_{ij}^{(0)}=\delta_{ij}$.

Introduce the $n \times n$ matrix $T(u)$ whose ij-th entry is the series

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \cdots \in \mathrm{Y}(\mathfrak{gl}_n)[[u^{-1}]].$$

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We can regard $T(u)$ as an element

$$T(u) = \sum_{i,j=1}^{n} e_{ij} \otimes t_{ij}(u) \in \operatorname{End} \mathbb{C}^n \otimes \mathrm{Y}(\mathfrak{gl}_n)[[u^{-1}]].$$

The defining relations of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$ can be written in the equivalent form

$$R(u-v)\,T_1(u)\,T_2(v) = T_2(v)\,T_1(u)\,R(u-v),$$

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Lemma. The matrix $M = T(u)\,e^{-\partial_u}$ is a Manin matrix.

q-Manin matrices

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A. Chervov, G. Falqui, V. Rubtsov and A. Silantyev, Algebraic properties of Manin matrices II: q-analogues and integrable systems, Adv. in Appl. Math. **60** (2014), 25–89.

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We will assume that $q \in \mathbb{C}^\times$.

A 2×2 matrix is q-Manin if the elements x' and y' defined by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

satisfy $y'x' = qx'y'$.

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Using $yx = qxy$, we get

$$(cx + dy)(ax + by) = q\,(ax + by)(cx + dy).$$

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$$ad - da = q^{-1}cb - qbc.$$

Definition. An $n \times n$ matrix M over an associative algebra $\mathcal{A}$ is a q-Manin matrix if all its 2×2 submatrices are Manin matrices:

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Determinants

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The q-column-determinant of a q-Manin matrix M is defined by

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In particular,

$$\mathrm{cdet}_q \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\,d - q^{-1}\,c\,b.$$

q-Deformed action of $\mathfrak{S}_k$

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The action of the symmetric group $\mathfrak{S}_k$ on the space $(\mathbb{C}^N)^{\otimes k}$ can be defined by setting $s_a \mapsto P^q_{s_a} := P^q_{a\,a+1}$, where s_a denotes the transposition $(a\;a+1)$ and P^q is the q-permutation operator

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This operator is an involution: $(P^q)^2 = 1$. Equivalently,

$$P^q(e_i \otimes e_j) = \begin{cases} q\, e_j \otimes e_i & \text{if} \quad i < j, \\ q^{-1}\, e_j \otimes e_i & \text{if} \quad i > j, \\ e_j \otimes e_i & \text{if} \quad i = j. \end{cases}$$

If $s = s_{a_1} \ldots s_{a_l}$ is a reduced decomposition of an element $s \in \mathfrak{S}_k$, we set $P^q_s = P^q_{s_{a_1}} \ldots P^q_{s_{a_l}}$.

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Warning. In general, $P^q_{(a\,b)} \neq P^q_{a\,b}$.

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Denote by $H^{(k)}$ and $A^{(k)}$ the q-symmetrizer and q-anti-symmetrizer:

$$H^{(k)} = \frac{1}{k!} \sum_{s \in \mathfrak{S}_k} P^q_s$$

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Key Lemma. M is a q-Manin matrix, if and only if

$$(1 - P^q)M_1 M_2 (1 + P^q) = 0.$$

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Claim. All the properties of Manin matrices have their natural q-analogues.

Super-Manin matrices

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P. H. Hai, B. Kriegk and M. Lorenz, N-homogeneous superalgebras, J. Noncommut. Geom. **2** (2008), 1–51.

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A. I. Molev and E. Ragoucy, The MacMahon Master Theorem for right quantum superalgebras and higher Sugawara operators for $\widehat{\mathfrak{gl}}_{m|n}$, Moscow Math. J. **14** (2014), 83–119.

We let $\mathbb{C}^{m|n}$ denote the $\mathbb{Z}_2$-graded vector space with the basis $e_1, \dots, e_{m+n}$ such that the degree (or parity) of e_i is 0 for $i = 1, \dots, m$ and is 1 for $i = m+1, \dots, m+n$.

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Set $\bar{\imath} = 0$ for $1 \leqslant i \leqslant m$ and $\bar{\imath} = 1$ for $m+1 \leqslant i \leqslant m+n$.

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We will consider superalgebras which are $\mathbb{Z}_2$-graded (associative) algebras $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$.

In particular, $\mathrm{End}\,\mathbb{C}^{m|n}$ is a superalgebra with the $\mathbb{Z}_2$-grading given by setting the parity of e_{ij} to be $\bar{\imath} + \bar{\jmath}$.

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We will consider even $(m+n) \times (m+n)$ matrices $Z = [z_{ij}]$ over a superalgebra $\mathcal{A}$ so that the (i,j) entry z_{ij} of Z has parity $\bar{\imath} + \bar{\jmath}$.

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Such a matrix Z will be identified with the element

$$Z = \sum_{i,j=1}^{m+n} e_{ij} \otimes z_{ij} (-1)^{\bar{\imath}\bar{\jmath}+\bar{\jmath}} \in \mathrm{End}\,\mathbb{C}^{m|n} \otimes \mathcal{A}.$$

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The signs are necessary because of the sign rule

$$(x\otimes y)(x'\otimes y')=(xx'\otimes yy')\,(-1)^{\deg y\,\deg x'}.$$

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For each $a\in\{1,\ldots,k\}$ the element Z_a of this superalgebra is defined by the formula

$$Z_a=\sum_{i,j=1}^{m+n}1^{\otimes(a-1)}\otimes e_{ij}\otimes 1^{\otimes(k-a)}\otimes z_{ij}\,(-1)^{\bar{\imath}\bar{\jmath}+\bar{\jmath}}.$$

The supertrace is the linear map

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The partial supertrace str_a acts as the supertrace map on the a-th copy of $\mathrm{End}\,\mathbb{C}^{m|n}$ and is the identity map on all the remaining copies.

Using the natural action of $\mathfrak{S}_k$ on $(\mathbb{C}^{m|n})^{\otimes k}$ we represent any permutation $\sigma \in \mathfrak{S}_k$ as an element P_σ of the superalgebra $\text{End}\,(\mathbb{C}^{m|n})^{\otimes k}$.

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In particular, the transposition $(a\,b)$ with $a < b$ corresponds to the element

$$P_{ab} = \sum_{i,j=1}^{m+n} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{ji} \otimes 1^{\otimes(k-b)} \, (-1)^{\bar{\jmath}},$$

which allows one to determine P_σ by writing an arbitrary $\sigma \in \mathfrak{S}_k$ as a product of transpositions.

Definition. An even matrix $Z = [z_{ij}]$ with entries in a superalgebra $\mathcal{A}$ is a Manin matrix, if

$$(1 - P_{12})\, Z_1 Z_2\, (1 + P_{12}) = 0$$

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Explicitly, the relations have the form

$$[z_{ij}, z_{kl}] = [z_{kj}, z_{il}](-1)^{\bar{\imath}\bar{\jmath}+\bar{\imath}\bar{k}+\bar{\jmath}\bar{k}}$$

where $[x, y] = xy - yx(-1)^{\deg x \deg y}$ is the super-commutator.

MacMahon Master Theorem

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Set

$$\text{Ferm} = 1 + \sum_{k=1}^{\infty} (-1)^k \operatorname{tr} A^{(k)} Z_1 \ldots Z_k,$$

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Theorem [MR 2014].

If Z is a Manin matrix, then

$$\mathrm{Bos} \times \mathrm{Ferm} = 1.$$

Berezinian

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The Berezinian of Z is defined by the formula

$$\begin{aligned}\operatorname{Ber} Z = &\sum_{\sigma\in\mathfrak{S}_m} \operatorname{sgn}\sigma \cdot z_{\sigma(1)1}\dots z_{\sigma(m)m}\\ &\times \sum_{\tau\in\mathfrak{S}_n} \operatorname{sgn}\tau \cdot z'_{m+1,m+\tau(1)}\dots z'_{m+n,m+\tau(n)}.\end{aligned}$$

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If $\mathcal{A}$ is supercommutative, then

$$\operatorname{Ber}(XY) = \operatorname{Ber} X \cdot \operatorname{Ber} Y.$$

Theorem. If Z is a Manin matrix, then

$$\operatorname{Ber}(1+uZ)=\sum_{k=0}^{\infty}u^k\,\operatorname{str}A^{(k)}Z_1\dots Z_k,$$

$$\big[\operatorname{Ber}(1-uZ)\big]^{-1}=\sum_{k=0}^{\infty}u^k\,\operatorname{str}H^{(k)}Z_1\dots Z_k,$$

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The last formula provides the Newton identities.

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1) Consider the associative algebra $\mathcal{M}_{m|n}$ with $(m+n)^2$ generators z_{ij} subject to the defining relations

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4) Develop the theory of q-super-Manin matrices.

Further generalizations

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- Manin matrices of types B, C, D.

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