Vinberg's problem for classical Lie algebras

Alexander Molev

University of Sydney

Joint work with Oksana Yakimova

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The subalgebra of invariants is

$$\mathrm{S}(\mathfrak{g})^{\mathfrak{g}} = \{P \in \mathrm{S}(\mathfrak{g}) \mid Y \cdot P = 0 \quad \text{for all} \quad Y \in \mathfrak{g}\}.$$

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The subalgebra $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}} \subset \mathrm{S}(\mathfrak{g})$ is Poisson commutative.

Integrability problem: Extend $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ to a big Poisson commutative subalgebra of $\mathrm{S}(\mathfrak{g})$.

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Denote by $\overline{\mathcal{A}}_\mu$ the subalgebra of $\mathrm{S}(\mathfrak{g})$ generated by all the μ-shifts $P_{(i)}$ associated with all invariants $P \in \mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$.

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- Moreover, $\overline{\mathcal{A}}_\mu$ is a maximal Poisson commutative subalgebra of $\mathrm{S}(\mathfrak{g})$ [D. Panyushev and O. Yakimova, 2008].

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Is it possible to quantize the subalgebra $\overline{\mathcal{A}}_\mu$ of $\mathrm{S}(\mathfrak{g})$?

We would like to find a commutative subalgebra $\mathcal{A}_\mu$ of $\mathrm{U}(\mathfrak{g})$ (together with its free generators) such that $\mathrm{gr}\,\mathcal{A}_\mu = \overline{\mathcal{A}}_\mu$.

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In the classical types, a direct evaluation homomorphism from

the Olshanski twisted Yangians $\mathrm{Y}^{\mathrm{tw}}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g})$ can be used

[M. Nazarov and G. Olshanski, 1996].

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$$\rho: \ X t^{r} \mapsto X z^{r} + \delta_{r,-1}\, \mu(X),$$

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FFTL-conjecture (2010): $\mathrm{gr}\, \mathcal{A}_\mu = \overline{\mathcal{A}}_\mu$.

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In particular, this gives a vector space isomorphism

$$\omega : \mathrm{S}(\mathfrak{g})^{\mathfrak{g}} \to \mathrm{Z}(\mathfrak{g}).$$

Conjecture [A.M. and O. Yakimova, 2017].

There exist free generators $H_1, \ldots, H_n$ of the algebra $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ such that for any $\mu \in \mathfrak{g}^*$ the ω-images of their μ-shifts generate the algebra $\mathcal{A}_\mu$.

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If the algebra is commutative then $\mathrm{Det}_m(M)$ is the sum of all principal m-minors of M.

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where γ_k denotes the multiplicity of $k \in \{1, \ldots, N\}$ in the multiset $\{a_1, \ldots, a_m\}$.

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All coefficients Ψ_m of the series

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This follows by taking the Chevalley images.

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where we assume that $E_{ij} \in \mathrm{S}(\mathfrak{gl}_N)$ on the left

and $E_{ij} \in \mathrm{U}(\mathfrak{gl}_N)$ on the right.

Theorem [Tarasov, 2000, 2003; M., Yakimova, 2017].

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Proof. Use the generators of the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{gl}}_N)$ found by [A. Chervov and D. Talalaev, 2006] and [A. Chervov and A. M., 2009] to get explicit generators of $\mathcal{A}_\mu$.

Types B, C and D

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Define the orthogonal Lie algebra $\mathfrak{o}_N$ with $N = 2n$ and $N = 2n + 1$ and symplectic Lie algebra $\mathfrak{sp}_N$ with $N = 2n$ as subalgebras of $\mathfrak{gl}_N$ spanned by the elements F_{ij},

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We use the involution $i \mapsto i' = N - i + 1$ on the set $\{1, \dots, N\}$, and in the symplectic case set

$$\varepsilon_i = \begin{cases} 1 & \text{for} \quad i = 1, \dots, n \\ -1 & \text{for} \quad i = n + 1, \dots, 2n. \end{cases}$$

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The coefficients $\Phi_2, \Phi_4, \dots, \Phi_{2n}$ are free generators

of the algebra $\mathrm{S}(\mathfrak{sp}_{2n})^{\mathfrak{sp}_{2n}}$.

The invariants $\Psi_{2m} \in \mathrm{S}(\mathfrak{o}_N)^{\mathfrak{o}_N}$ are defined by

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For $N = 2n$ define the Pfaffian by

$$\mathrm{Pf}\, F = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \mathrm{sgn}\, \sigma \cdot F_{\sigma(1)\, \sigma(2)'} \dots F_{\sigma(2n-1)\, \sigma(2n)'}.$$

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For $\mu \in \mathfrak{g}^*$ set $\mu_{ij} = \mu(F_{ij})$ and consider the matrix

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Introduce another Young diagram by

$$\Pi = \alpha^{(1)} + \dots + \alpha^{(r)},$$

the sum is taken by rows.

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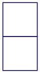
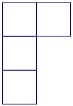
Example. Take $N = 6$ and let μ have two distinct eigenvalues with the associated Young diagrams





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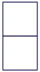

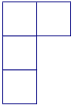

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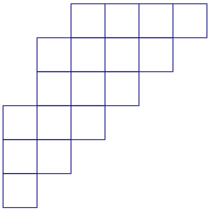

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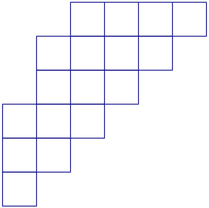

Thus we exclude $\Phi_{6\,5}$, $\Phi_{6\,4}$, $\Phi_{5\,4}$ and $\Phi_{4\,3}$.

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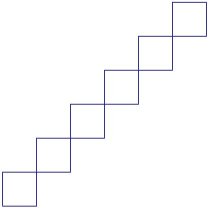

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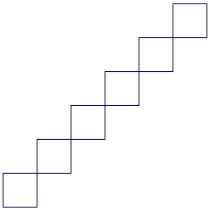

Thus $\mathcal{A}_\mu = \mathbb{C}\left[\Phi_{1\,0}, \ldots, \Phi_{N\,0}\right] = \mathrm{Z}(\mathfrak{gl}_N)$.

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In types A and C:

$\overline{\mathcal{A}}_\mu$ is a maximal Poisson commutative subalgebra of $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}_\mu}$.

Hence $\operatorname{gr} \mathcal{A}_\mu = \overline{\mathcal{A}}_\mu$.