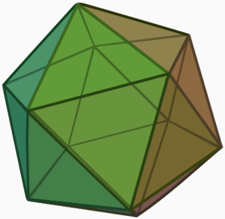
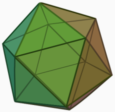
REPRESENTATION THEORY AND GEOMETRY

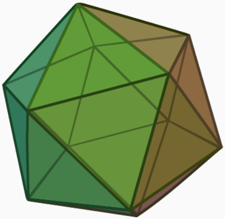
Geordie Williamson

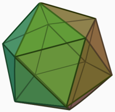
University of Sydney
http://www.maths.usyd.edu.au/u/geordie/ICM.pdf

REPRESENTATIONS



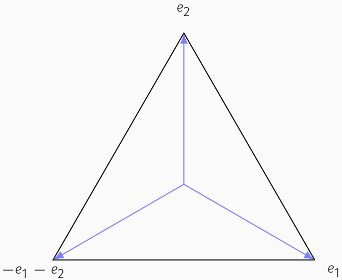
$\rightsquigarrow$ $\mathrm{Sym}\Bigg($  $\Bigg) \subset GL(\mathbb{R}^3).$



$\rightsquigarrow \quad \mathrm{Sym}\Bigg($  $\Bigg) \subset GL(\mathbb{R}^3).$

We obtain a representation of our group of symmetries

$$\rho : G \to GL(V).$$



$$\downarrow$$

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

WHY STUDY REPRESENTATIONS?

Symmetric group S_n

The problem of understanding

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Galois representations
The passage(s)

$$\{\text{varieties}/\mathbb{Q}\} \longrightarrow \{\text{Galois representations}\}$$

is one of the most powerful tools of modern number theory.

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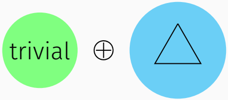

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A representation is semi-simple if it is isomorphic to a direct sum of simple representations.

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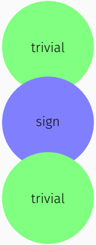
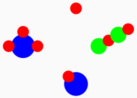
We write ("Grothendieck group", "multiplicities")

$$[\mathbb{F}_3^3] = [L] + [H/L] + [\mathbb{F}_3^3/H] = 2[\mathrm{trivial}] + [\mathrm{sign}].$$



representations ↔ “matter”



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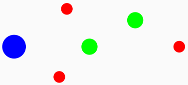

SIMPLE REPRESENTATIONS

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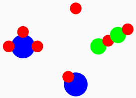

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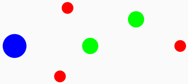

We search for a classification (“periodic table”), character formulas (“mass”, “number of neutrons”), …

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Related situations: non-compact Lie groups, p-adic groups...

COMMON FEATURES

representation theory

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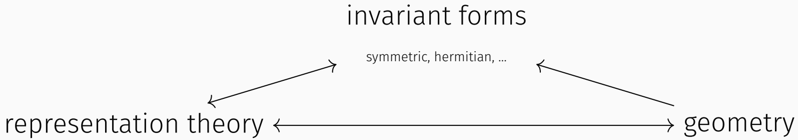
representation theory $\longleftrightarrow$ geometry


invariant forms
symmetric, hermitian, ...
representation theory
geometry

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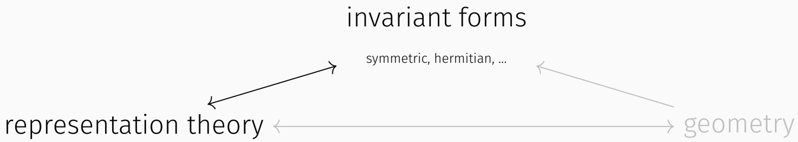
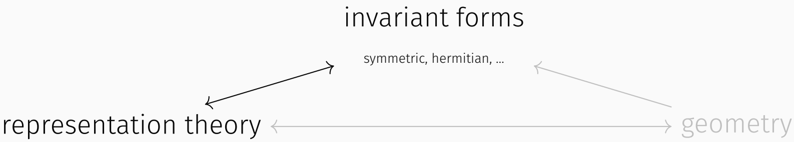


COMMON FEATURES

Related feature:
(hidden) semi-simplicity



Related feature:
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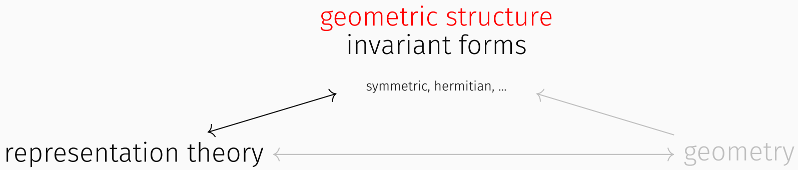
We do not assume that our forms are positive definite; signature plays an important role throughout.



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THE SEMI-SIMPLE WORLD

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Observation 2: Any representation of G admits a positive-definite geometric structure.

Take a positive-definite geometric structure $\langle -, - \rangle$ on V. Then

$$\langle v, w \rangle_G := \frac{1}{|G|} \sum \langle gv, gw \rangle$$

defines a positive-definite and G-invariant geometric structure.

GEOMETRIC STRUCTURE

Example of “semi-simplicity via introduction of geometric structure”.

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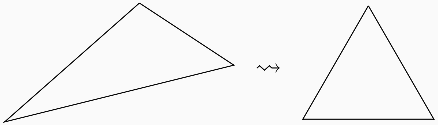
This is an example of "unicity of geometric structure".



Consider a compact Lie group K, e.g. S^1 or SU_2 or a finite group.

Weyl generalised these observations to K, with sum replaced by integral:

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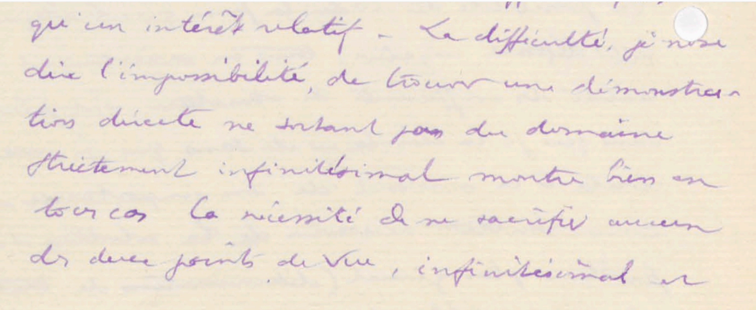
Existence and uniqueness of geometric structure still holds.

Élie Cartan to Hermann Weyl, 28 of March 1925:



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"...the difficulty, I dare not say the impossibility, of finding a proof which does not leave the strictly infinitesimal domain shows the necessity of not sacrificing either point of view ..."

An algebraic ("infinitesimal") proof took 10 years, and involves the Casimir element (arises from an invariant form called the trace form).

EXTENDED EXAMPLE: su_2 AND $\mathfrak{sl}_2$

$$SU_2 = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, AA^* = \mathrm{id}, \det A = 1 \right\} = \begin{array}{c} \text{unit} \\ \text{quaternions.} \end{array}$$

$$\mathrm{Lie}(SU_2)_{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C}) = \mathbb{C} \underset{\substack{\| \\ f}}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} \oplus \mathbb{C} \underset{\substack{\| \\ h}}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \oplus \mathbb{C} \underset{\substack{\| \\ e}}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$$

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"I don't think it is the representations themselves, but the groups. I find SU_2, SL_2, S_n etc. amazing and beautiful animals (if I have a favourite, it is SU_2), but will probably never really understand them. I might someday understand their linear shadows though..."

– Quindici

SU_2 acts on its “natural representation”:

$$\mathbb{C}^2 = \mathbb{C}\begin{pmatrix}0\\1\end{pmatrix} \oplus \mathbb{C}\begin{pmatrix}1\\0\end{pmatrix} = \mathbb{C}Y \oplus \mathbb{C}X.$$

For any $m \geqslant 0$, SU_2 acts naturally on homogenous polynomials in X, Y of degree m:

$$L_m := \mathbb{C}Y^m \oplus \mathbb{C}Y^{m-1}X \oplus \cdots \oplus \mathbb{C}Y^m X^{m-1} \oplus \mathbb{C}X^m.$$

The L_m for $m \geqslant 0$ are all irreducible representations of SU_2.

“spherical harmonics”, “quantum mechanics”.

Differentiate to get representation of the (complexified) Lie algebra

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Action on L_m (here $m = 5$):

$\mathbb{C}Y^5$ $\quad$ $\mathbb{C}Y^4X^1$ $\quad$ $\mathbb{C}Y^3X^2$ $\quad$ $\mathbb{C}Y^2X^3$ $\quad$ $\mathbb{C}YX^4$ $\quad$ $\mathbb{C}X^5$

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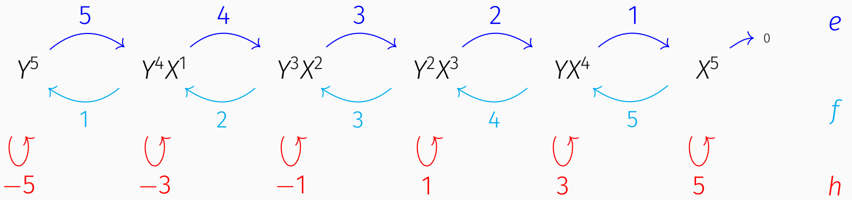

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For $\mathfrak{sl}_2(\mathbb{C})$ they depend on a parameter $\lambda \in \mathbb{C}$.

As vector spaces:

$$\Delta_\lambda = \bigoplus_{i \in \mathbb{Z}_{\geqslant 0}} \mathbb{C} v_i$$

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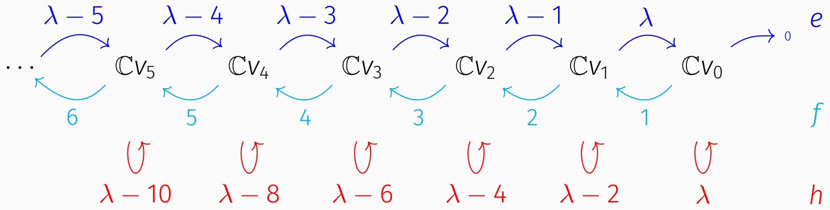
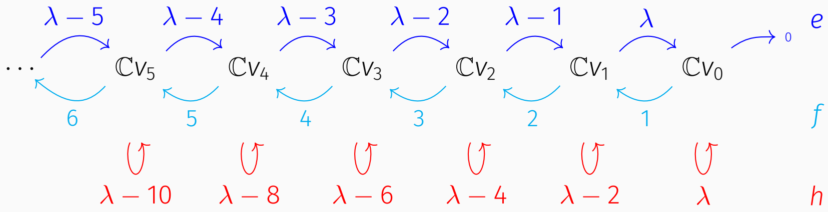
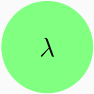
The Verma module Δ_λ determined by $\lambda \in \mathbb{C}$:

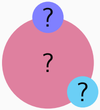




$\lambda \neq 0, 1, 2, 3, \ldots \quad : \quad \Delta_\lambda$ is simple, call it L_λ.

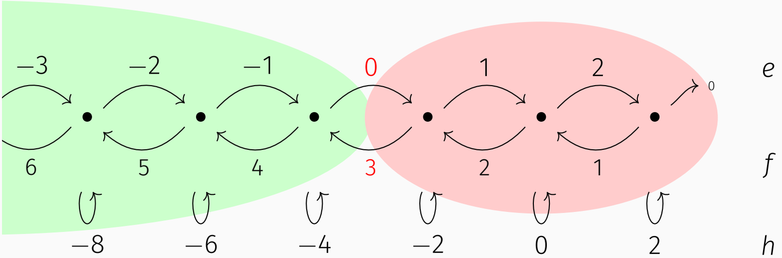


$\lambda = 0, 1, 2, 3, \ldots \quad : \quad \Delta_\lambda$ is not simple.



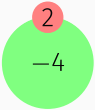
STRUCTURE OF VERMA MODULES

Example $\lambda = 2$



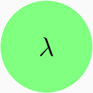
We have a subrepresentation isomorphic to Δ_{-4}, and

$$\Delta_2/\Delta_{-4} \cong L_2$$

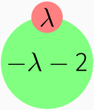


(L_2 is our simple finite-dimensional representation from earlier.)

$\lambda \neq 0, 1, 2, 3, \ldots$: Δ_λ is simple and infinite-dimensional.



$\lambda = 0, 1, 2, 3, \ldots$: Δ_λ is not simple.



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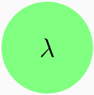

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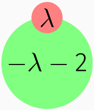
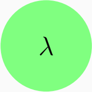


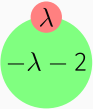
Summary:

STRUCTURE OF VERMA MODULES

$\lambda \neq 0, 1, 2, 3, \dots$: Δ_λ is simple and infinite-dimensional.



$\lambda = 0, 1, 2, 3, \dots$: Δ_λ is not simple.



Summary:

(a) A single family of representations (Verma modules) produces all simple finite-dimensional representations.

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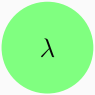

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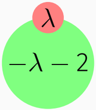

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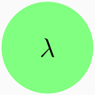

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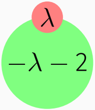

Summary:

(a) A single family of representations (Verma modules) produces all simple finite-dimensional representations.
(b) We get new infinite-dimensional simple representations.
(c) The structure of Verma modules varies (subtly) based on the parameter.

Kazhdan–Lusztig conjecture

$\mathfrak{g}$ is a complex semi-simple Lie algebra.

$\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra.

W the Weyl group, which acts on $\mathfrak{h}$ as a reflection group.

Example

$\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) = \{ n \times n \text{ matrices } X \mid \mathrm{tr} X = 0\}$.
$\mathfrak{h}$ = diagonal matrices $\subset \mathfrak{sl}_n(\mathbb{C})$
$W = S_n$ acting on $\mathfrak{h}$ via permutations.

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Motivation

We think of the finite group W as being the skeleton of $\mathfrak{g}$.

We try to answer questions about $\mathfrak{g}$ in terms of W.

$\mathfrak{g}$ is a complex semi-simple Lie algebra.

“weight” $\lambda \in \mathfrak{h}^*$ $\rightsquigarrow$ *Verma module* Δ_λ.

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"weight" $\lambda \in \mathfrak{h}^* \rightsquigarrow$ *Verma module* Δ_λ.

Δ_λ has unique simple quotient $\Delta_\lambda \twoheadrightarrow L_\lambda$

L_λ is called a simple highest weight module.

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If $\lambda \neq 0, 1, \ldots, L_\lambda = \Delta_\lambda$ is infinite dimensional.
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Basic problem

Describe the structure of Δ_λ.
Which simple modules occur with which multiplicity?

Δ_λ : Verma module. L_λ : simple highest weight module.

Kazhdan-Lusztig conjecture (1979)

$$[\Delta_\lambda] = \sum_{\mu} P_{\lambda,\mu}(1)[L_\mu].$$

Here $P_{\lambda,\mu} \in \mathbb{Z}[v]$ is a Kazhdan-Lusztig polynomial.

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KAZHDAN-LUSZTIG CONJECTURE

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Kazhdan-Lusztig conjecture (1979)

$$[\Delta_{x\cdot 0}] = \sum_{y \in W} P_{x\cdot 0, y\cdot 0}(1)[L_{y\cdot 0}].$$

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(a) New paradigm in representation theory.
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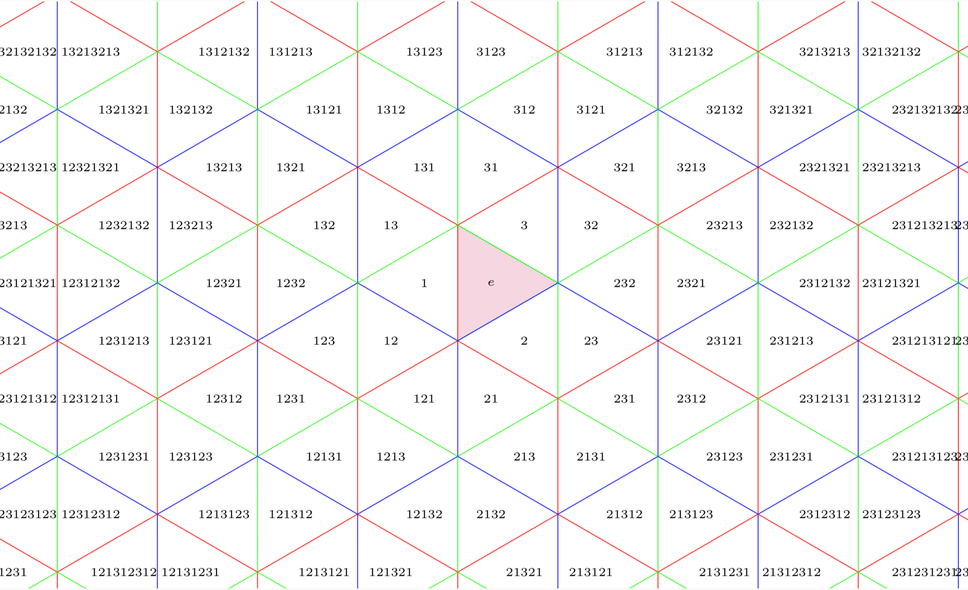
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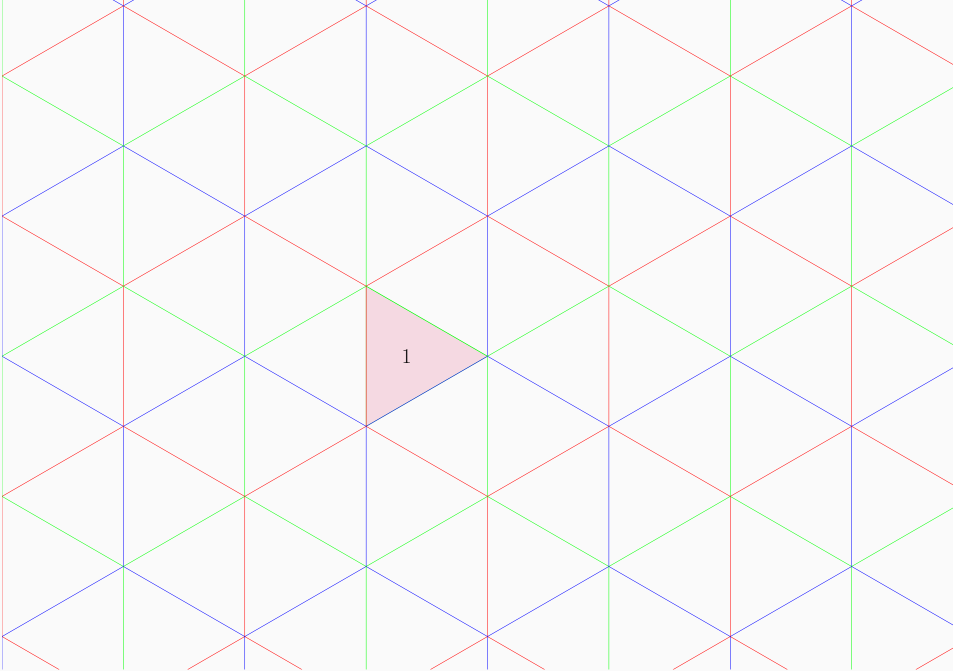
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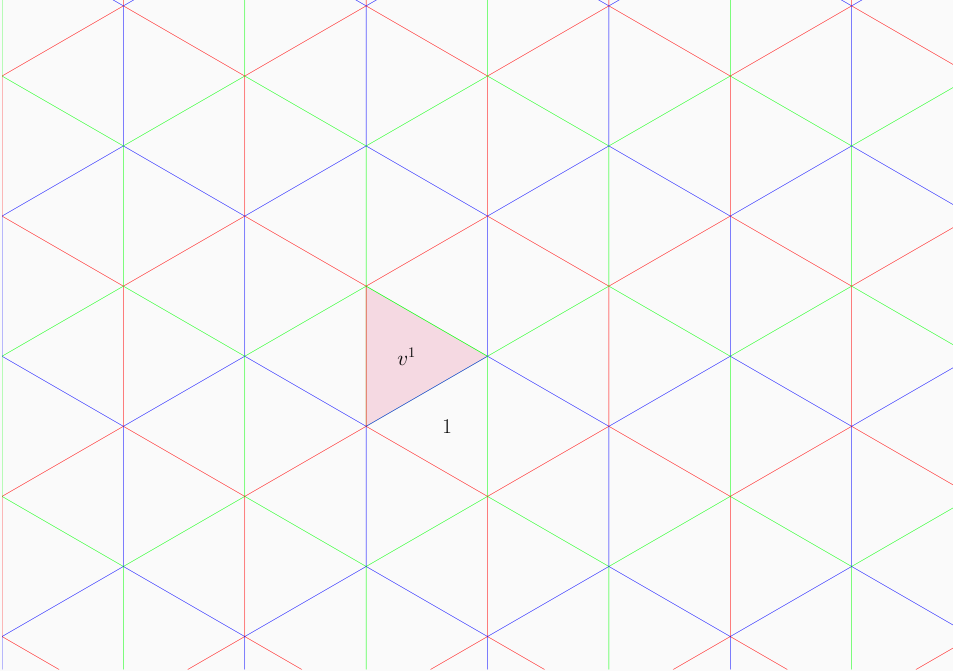
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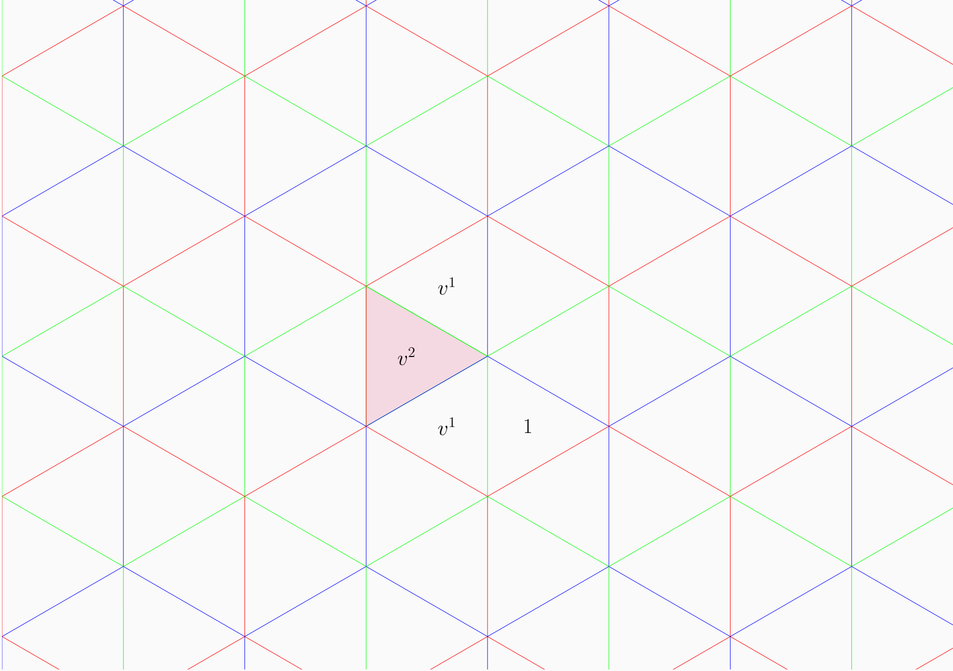
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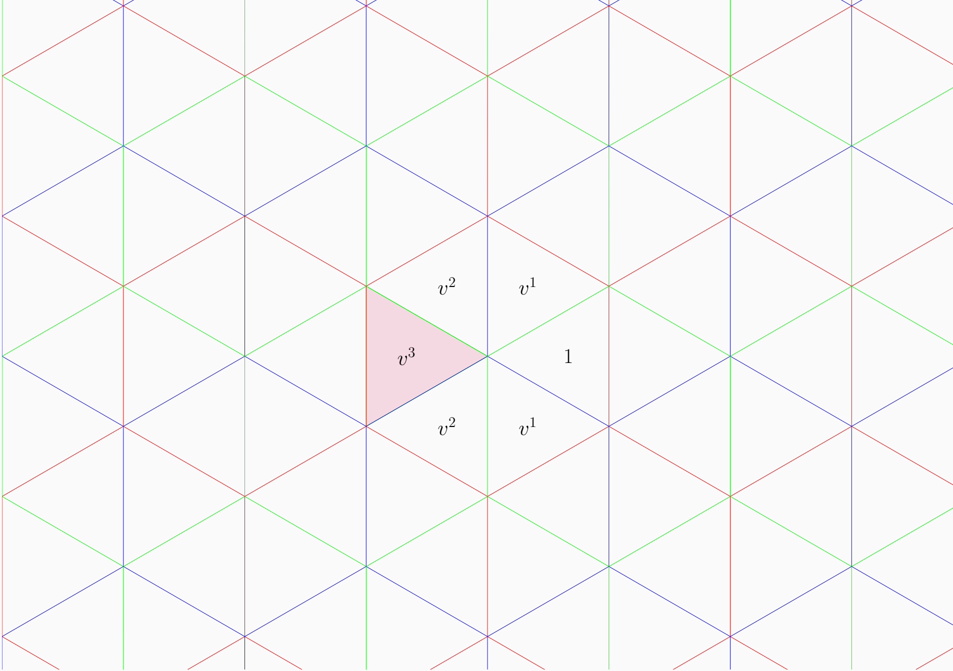
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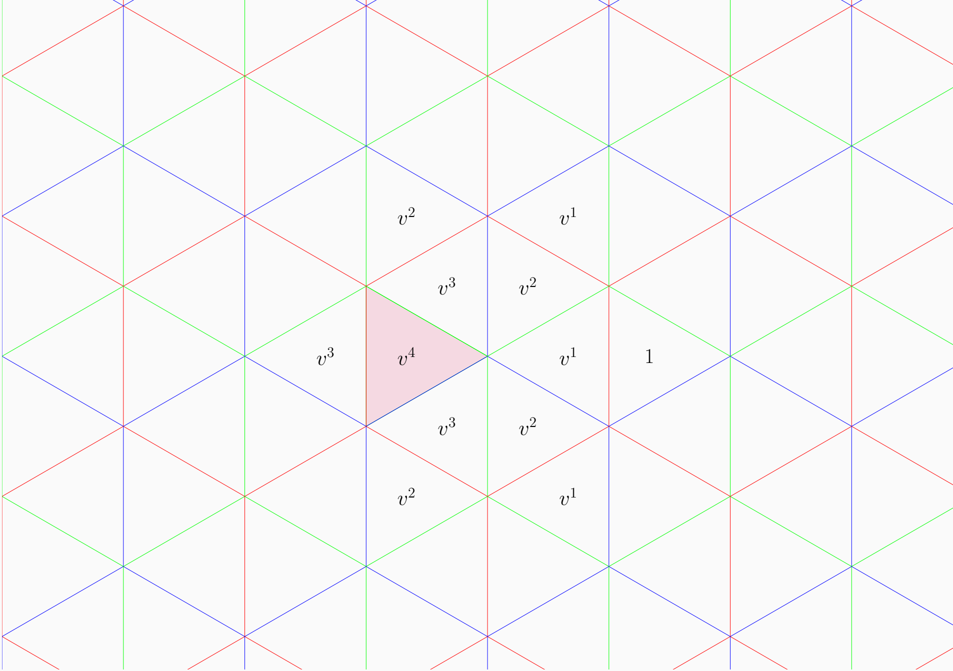
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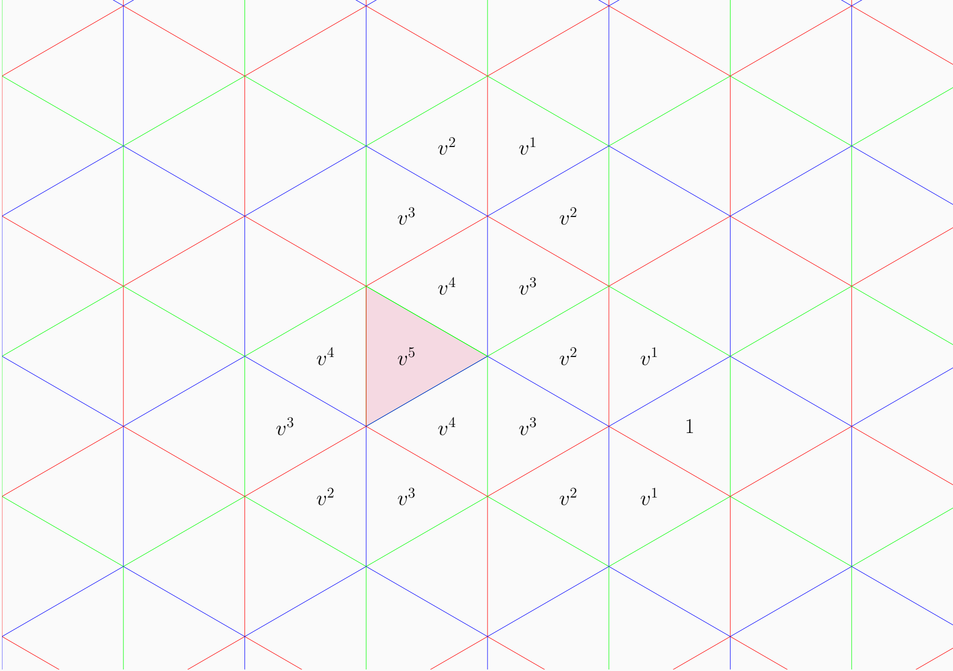

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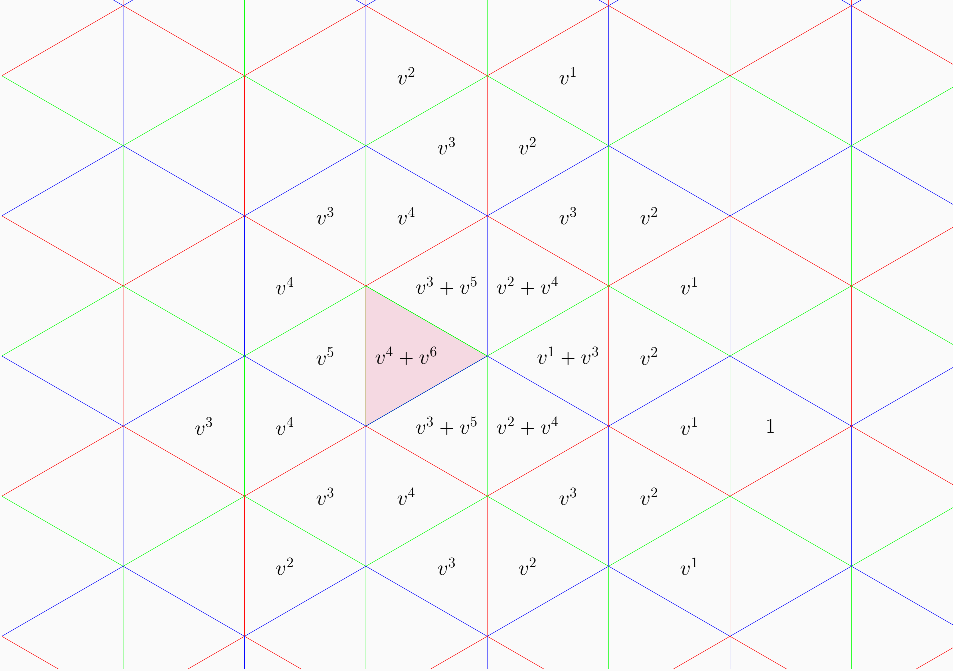
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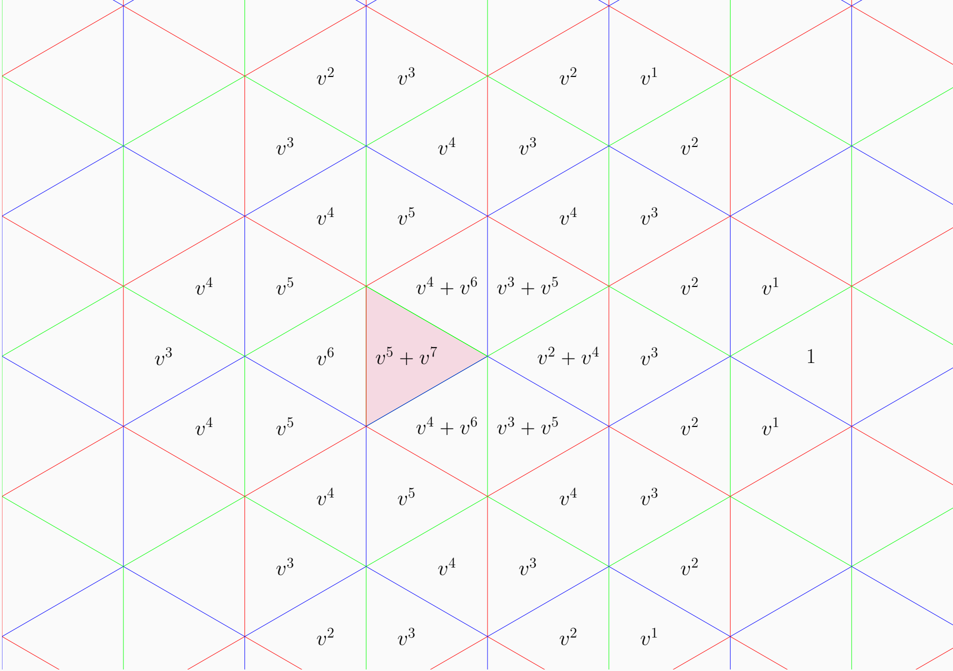

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v^4 v^5 v^4+v^6 v^3+v^5 v^2 v^1
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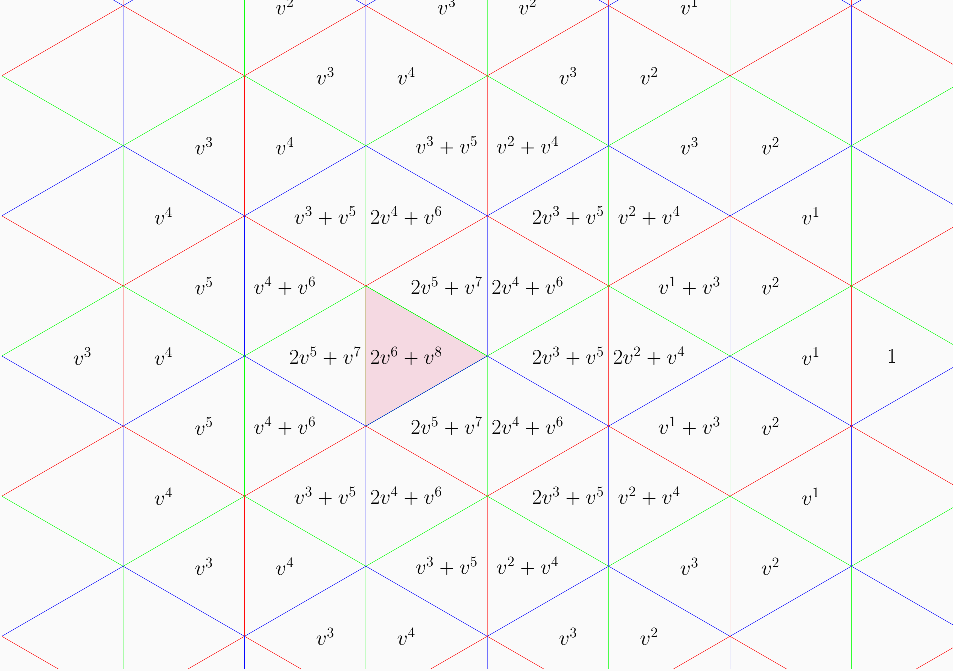

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v^5 $v^4 + v^6$ $2v^5 + v^7$ $2v^4 + v^6$ $v^1 + v^3$ v^2
v^3 v^4 $2v^5 + v^7$ $2v^6 + v^8$ $2v^3 + v^5$ $2v^2 + v^4$ v^1 1
v^5 $v^4 + v^6$ $2v^5 + v^7$ $2v^4 + v^6$ $v^1 + v^3$ v^2
v^4 $v^3 + v^5$ $2v^4 + v^6$ $2v^3 + v^5$ $v^2 + v^4$ v^1
v^3 v^4 $v^3 + v^5$ $v^2 + v^4$ v^3 v^2
v^3 v^4 v^3 v^2

	Geometric proof ('80s)
Kazhdan-Lusztig conjecture (multiplicity $= P_{y,x}(1)$)	Brylinsky-Kashiwara Beilinson-Bernstein

POTTED HISTORY

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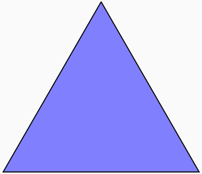
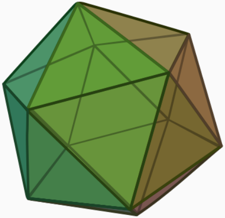
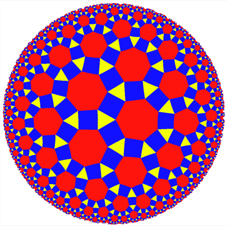
Geometric proofs: D-modules, perverse sheaves, weights...

Algebraic proofs: “shadows of Hodge theory”,
i.e. invariant forms (“geometric structures”)
still satisfying Poincaré duality, Hard Lefschetz, Hodge-Riemann

SHADOWS OF HODGE THEORY







Weyl groups $\subset$ Real reflection groups $\subset$ Coxeter groups

Let W denote a real reflection group acting on $\mathfrak{h}_{\mathbb{R}}$.

THE COINVARIANT RING

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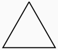
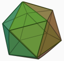
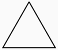
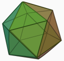
Example

$\mathrm{Sym}\Big($  $\Big) \circlearrowright \mathbb{R}^2$ or $\mathrm{Sym}\Big($  $\Big) \circlearrowright \mathbb{R}^3$.

Let W denote a real reflection group acting on $\mathfrak{h}_{\mathbb{R}}$.

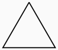
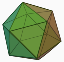
Example

$\mathrm{Sym}\Big($  $\Big) \circlearrowright \mathbb{R}^2$ or $\mathrm{Sym}\Big($  $\Big) \circlearrowright \mathbb{R}^3$.

Let R denote the polynomial functions on $\mathfrak{h}_{\mathbb{R}}$. We view R as graded with $\mathfrak{h}_{\mathbb{R}}^*$ in degree 2.

Let W denote a real reflection group acting on $\mathfrak{h}_{\mathbb{R}}$.

Example

$\mathrm{Sym}\Big($  $\Big) \circlearrowright \mathbb{R}^2$ or $\mathrm{Sym}\Big($  $\Big) \circlearrowright \mathbb{R}^3$.

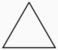
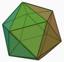
Let R denote the polynomial functions on $\mathfrak{h}_{\mathbb{R}}$. We view R as graded with $\mathfrak{h}^*_{\mathbb{R}}$ in degree 2.

Let R^W_+ denote the *W*-invariants of positive degree. Set

$$H := R/(R^W_+).$$

Let W denote a real reflection group acting on $\mathfrak{h}_{\mathbb{R}}$.

Example

$\mathrm{Sym}\Big($  $\Big) \circlearrowright \mathbb{R}^2$ or $\mathrm{Sym}\Big($  $\Big) \circlearrowright \mathbb{R}^3$.

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Let R_+^W denote the *W-invariants* of positive degree. Set

$$H := R/(R_+^W).$$

Remark

If W is the Weyl group of a complex semi-simple Lie algebra $\mathfrak{g}$, then H is isomorphic to the cohomology of the flag variety of $\mathfrak{g}$ (the "Borel isomorphism").

$$H := R/(R_+^W)$$

$d :=$ number of reflecting hyperplanes in $\mathfrak{h}_{\mathbb{R}}$

"complex dimension of flag variety"

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$$d := \text{number of reflecting hyperplanes in } \mathfrak{h}_{\mathbb{R}}$$

"complex dimension of flag variety"

There exists a unique (up to scalar) bilinear form

$$\langle -, - \rangle : H^{d-\bullet} \times H^{d+\bullet} \to \mathbb{R}$$

satisfying $\langle \gamma c, c' \rangle = \langle c, \gamma c' \rangle$ for all $\gamma, c, c' \in H$ (the invariant form).

$$H := R/(R^W_+)$$
$$d := \text{number of reflecting hyperplanes in } \mathfrak{h}_{\mathbb{R}}$$
"complex dimension of flag variety"

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satisfying $\langle \gamma c, c' \rangle = \langle c, \gamma c' \rangle$ for all $\gamma, c, c' \in H$ (the invariant form).

Remark

$\langle -, - \rangle$ is the analogue of the intersection form on cohomology.

There exists an open cone $K \subset \mathfrak{h}^*_{\mathbb{R}}$ (“Kähler cone”).

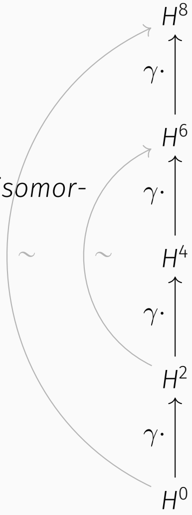
There exists an open cone $K \subset \mathfrak{h}_{\mathbb{R}}^*$ ("Kähler cone").

Theorem (toy model)

For all $\gamma \in K$ and $i \geqslant 0$:

(a) (Hard Lefschetz) Multiplication by γ^i induces an isomorphism

$$H^{d-i} \xrightarrow{\sim} H^{d+i}$$



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(a) (Hard Lefschetz) Multiplication by γ^i induces an isomorphism

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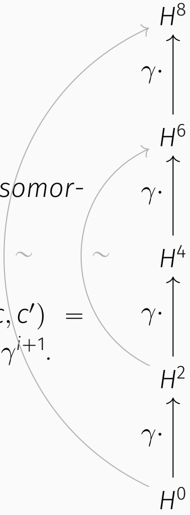
(b) (Hodge-Riemann bilinear relations) The form $(c, c') = \langle c, \gamma^i c' \rangle$ on H^{d-i} is $(-1)^?$-definite on the kernel of γ^{i+1}.



There exists an open cone $K \subset \mathfrak{h}^*_{\mathbb{R}}$ (“Kähler cone”).

Theorem (toy model)

For all $\gamma \in K$ and $i \geqslant 0$:

(a) (Hard Lefschetz) Multiplication by γ^i induces an isomorphism

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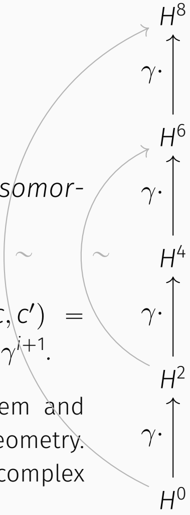
(b) (Hodge-Riemann bilinear relations) The form $(c, c') = \langle c, \gamma^i c' \rangle$ on H^{d-i} is $(-1)^?$-definite on the kernel of γ^{i+1}.

The theorem is identical to the hard Lefschetz theorem and Hodge-Riemann bilinear relations in complex algebraic geometry. In the Weyl group case the theorem can be deduced from complex algebraic geometry, but not in general.



In 1990 Soergel defined graded H-modules H_w for all $w \in W$. Today they are known as "Soergel modules". In the Weyl group case, he proved that the Kazhdan-Lusztig conjecture is equivalent to

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We provided an algebraic proof of (1) as a consequence of

Theorem (Elias-W.)

The hard Lefschetz and Hodge-Riemann relations hold for H_w.

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Diagrammatic algebra crucial to calculate and discover correct statements.

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A mystery for the 21^{st} century?

Similar structures arise in the theory of non-rational polytopes (due to McMullen, Braden-Lunts, Karu, ...) and in recent work of Apridisato-Huh-Katz on matroids. Why?

MODULAR REPRESENTATIONS

There are analogies between infinite-dimensional representations of Lie algebras, and modular representations of algebraic groups. The analogue of the Kazhdan-Lusztig conjecture in this setting is the Lusztig conjecture (1980).

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Invariant forms (now defined over the integers) still play a decisive role in the theory.

Lusztig conjecture (1980)

$$[\hat{\Delta}_A] = \sum_B q_{A,B}(1)[\hat{L}_B]$$

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(d) False for primes growing exponentially in the rank (W. 2014, following He-W. 2013), e.g. false for $p = 470\,858\,183$ for SL_{100}.

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${}^p q_{A,B}$ are computable via diagrammatic algebra + computer.

BILLIARDS CONJECTURE

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The conjecture predicts that these numbers are given by a "discrete dynamical system"...

Billiards and tilting characters:
`https://www.youtube.com/watch?v=Ru0Zys1Vvq4`