A REMARK ON 4-MANIFOLDS WITH ZERO ENTROPY METRICS

J. A. HILLMAN

Abstract. We sharpen a recent result of Paternain and Petean by showing that a 4-manifold which admits a Riemannian metric with zero topological entropy either has a finite covering space homeomorphic to $S^3 \times S^1$ or $S^2 \times T$ or is aspherical.

Paternain and Petean showed that if a 4-manifold admits a Riemannian metric with zero topological entropy then its universal covering space has the rational homotopy type of S^2 or S^3 or of a point [7]. They show also that if $\pi_1(M)$ has polynomial growth then M has a finite covering space homeomorphic to $S^3 \times S^1$, $S^2 \times T$ or a nilmanifold. We may sharpen this result as follows.

Theorem. *Let M be a 4-manifold that admits a Riemannian metric with zero topological entropy. Then either M has a finite covering space homeomorphic to $S^3 \times S^1$ or $S^2 \times T$ or it is aspherical. In the latter case, if $\pi = \pi_1(M)$ has a subgroup σ of finite index with $\beta_1(\sigma) \geq 2$ then it is virtually nilpotent and M is homeomorphic to an infranilmanifold.*

Proof. We may clearly assume that M is orientable in order to prove the first assertion. The zero entropy hypothesis implies that π has subexponential growth [3]. Moreover $\chi(M) = 0$, by Theorem 4.1 of [7], and so π maps onto Z (cf. Lemma 3.14 of [5]). Hence π is an HNN extension $HNN(B; \phi : I \to J)$ with finitely generated base B [1]. Such an HNN extension has exponential growth unless $I = J = B$ (see page 196 of [2]). Hence $\pi \cong B \rtimes Z$. If B is finite or has two ends then π is virtually Z or Z^2, and the result follows from Theorems 10.10 and 11.1 of [5]. Otherwise B has one end (since it has subexponential growth), and so M is aspherical and B is a PD_3^+-group, by Corollary 6.3 of [6].

If π has a subgroup σ of finite index with $\beta_1(\sigma) \geq 2$ then $B_1 = B \cap \sigma$ has finite index in B and maps onto Z. Since B_1 is FP_2 it is an HNN extension with finitely generated base C [1], and since it has subexponential growth it is a semidirect product $B_1 \cong C \rtimes Z$ [2]. Hence C must be a PD_2^+-group, by Corollary 6.1 of [6]. Moreover $C \cong Z^2$, since it has subexponential growth. Therefore π is virtually solvable, and hence virtually nilpotent [4]. The final assertion follows from Theorem 8.1 of [5]. □

Must π be virtually nilpotent in all cases? This would be so if every PD_3-group has a subgroup of finite index with infinite abelianization.

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School of Mathematics and Statistics,, University of Sydney, NSW 2006, AUSTRALIA
E-mail address: jonh@maths.usyd.edu.au