A new approach to assignments and examinations in mathematics courses

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Abstract: *A combination of assignments and examinations avoids unethical behaviour whilst providing favourable learning outcomes and accurate grading and ranking of students.*

Introduction

In traditional mathematics courses students complete assignments which are marked, returned with comments, and contribute some proportion, X per cent say, to their final grades. The number X is usually very low to offset problems with cheating. The benefit of getting students actively involved by completing tough assignments is seen as outweighing the effort disproportionate to the low percentage of credit and disadvantages associated with cheating. Students on the whole are diligent in handing in complete assignments. It is common for students to be encouraged to collaborate, but asked to write up their findings independently. Certainly anecdotal evidence is strong that for a substantial number of students, particularly in advanced streams, the activity is beneficial and the feedback useful. But there is also strong anecdotal and actual evidence that the traditional assignment system corrupts the final rankings in the pass to credit range in large classes, which may damage the integrity of courses and the ability of students to cope in subsequent years. Many marked assignments remain unclaimed (and markers' comments unread) in collection boxes. The breakdown of marks shows discrepancies between performances in exams and assignments, and that a disturbingly high number of students rely on the 'assignment cushion' to limp over the passing mark threshold, possibly without having developed genuine competence. Important questions should be considered: are traditional methods an intelligent use of our dwindling resources? To what extent do they facilitate learning? Can we adjust these methods to help students learn more and create fairer rankings and better predictors of how students will cope in subsequent years? Over the last decade, the author has integrated assignments and examinations in large second year courses. An example is given in this paper, where assignments did not formally count (so X equalled zero), but were self-assessed by students, and the content integrated with exam questions. The method evolved for pedagogical reasons and in response to dissatisfaction about widespread cheating in class surveys. The ideas in this paper also should be read in the wider context of vertical (as opposed to horizontal) teaching, a brief introduction to which can be gleaned, in the context of the Halmos Principle, from Easdown (2006a,b), and the need to address the passive/active interface in helping students become fluent 'drivers' of mathematics, as explained in Easdown (2006c,d).

Case study: Big-Oh and Big-Omega running times of algorithms

The materials described here come from the teaching of MATH2011 Topics in Discrete Mathematics, with an enrolment of 94 students in 2003. This course was designed in response to needs of students in Information Science and Engineering, and included an introduction to algorithm analysis, an exposition for which can be found in Easdown (2003) or Rosen (2003). In lectures, Big-Oh notation and worst case running times of algorithms were introduced and criteria given for comparing running times. Aspects of the theory were illustrated and applied in various contexts, such as sorting and searching, the divide-and-conquer paradigm, performing 'fast' arithmetic and finding convex hulls. Basics were drilled and details explored in the tutorial exercises and practice classes. This topic utilised about 3 teaching weeks and was regarded as the most difficult part of the course, and became the main focus of the Assignment (Figures 1 and 2). which introduced Big-Omega notation, the natural theoretical framework for describing lower bounds on running times, and transformation of algorithms. It lead students in steps towards the important result that the convex hull algorithm described in lectures is optimal. It was designed to

- engage students in an interesting, useful and challenging project, closely related to but not covered by the lecture and tutorial materials;
- encourage students to collaborate yet develop independent thinking skills, by reflecting on their work and correcting mistakes;
- facilitate an end-of-semester examination which fairly ranks students according to their overall competence, skill and perseverence in mastering discrete mathematics.

Figure 1 comprises the first two Assignment questions and a brief preamble explaining clearly that the material is examinable and solutions will be available. The first question defines Big-Omega, mirroring Big-Oh in lectures, and concludes with a result utilising limits (first year calculus). The second question uses determinants of matrices (first year linear algebra) applied to convexity of parabolas. These are ingredients for the final question of the Assignment, given in Figure 2, which culminates in the theorem that the running time of a convex hull algorithm is unavoidably at least $N \log N$ for an input of size N. The Assignment is intended to be rounded, purposeful and deeply satisfying to a student who figures it out. Even those that have not mastered all of the detail will have an appreciation of the role of discrete mathematics in theoretical computer science.

THE UNIVERSITY OF SYDNEY

MATH2011 TOPICS IN DISCRETE MATHEMATICS

Semester 1 **Assignment** 2003

This assignment is not to be handed in. However some parts of exam questions may relate to the assignment, so it is in your interests to attempt it, especially if you are aiming for a credit or higher. Solutions will be available from the webpage from Friday 4 June.

1. The Big-Oh notation is a convenient device for simplifying upper bounds. A useful device for dealing with lower bounds is the Big-Omega notation. Consider nonnegative functions $f(N)$ and $g(N)$ of a natural number N. We say

$$f(N) \text{ is } \Omega(g(N)) \text{ and write } f(N) = \Omega(g(N))$$

if there exist positive constants K and N_0 such that

$$f(N) \geq Kg(N) \quad \text{for all} \quad N \geq N_0 \,.$$

(a) Verify that $f(N) = \Omega(g(N))$ if and only if $g(N) = O(f(N))$.

(b) Show that $N^3 - 10N^2 + 10,000 = \Omega(N^3)$.

(c) Consider the following conditions, where K is a positive constant:

$$\text{(i)} \lim_{N\to\infty} \frac{f(N)}{g(N)} = \infty \quad \text{(ii)} \lim_{N\to\infty} \frac{f(N)}{g(N)} = 0 \quad \text{(iii)} \lim_{N\to\infty} \frac{f(N)}{g(N)} = K$$

Which of (iv) $f(N) = \Omega(g(N))$ (v) $g(N) = \Omega(f(N))$

(vi) $f(N) \neq \Omega(g(N))$ (vii) $g(N) \neq \Omega(f(N))$

are implied by each of (i), (ii), (iii)? (There is no need to supply proofs.)

(d) Assuming $\lim_{N\to\infty} \frac{g(N)+f(N)}{N \log N}$ exists, use limits to verify that if $f(N) = O(N)$ and $g(N)+f(N) = \Omega(N \log N)$ then $g(N) = \Omega(N \log N)$.

2. Let x, y, z be real numbers and put $A = \begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{bmatrix}$.

(a) Verify that $\det A = (y-x)(z-x)(z-y)$. Deduce that $\det A > 0$ if and only if either $x < y < z$ or $y < z < x$ or $z < x < y$.

(b) Now deduce that if $n \geq 3$ and $x_1, \ldots, x_n$ are positive real numbers then the points

$$(x_1, x_1^2),\ (x_2, x_2^2),\ \ldots,\ (x_n, x_n^2)$$

are vertices of a convex polygon in standard form if and only if

$$x_1 < x_2 < \ldots < x_n \,.$$

Figure 1. First two questions from the Assignment for MATH2011 in 2003

3. It is an important theorem in algorithm analysis that all sorting algorithms have running time

$$T(N) \;=\; \Omega(N \log N)\,.$$

(The proof uses decision trees, and will not be given in this course.) Hence it is impossible to improve on $O(N \log N)$ as the running time of sorting algorithms. Thus, for example, MERGESORT, whose running time is $O(N \log N)$, is an optimal sorting algorithm. The purpose of this exercise is to use the above theorem to prove an analogous result for convex hull algorithms.

Suppose $\mathcal{C}$ is a convex hull algorithm which takes as input a finite set X of points in the plane and outputs the list of vertices of the boundary of HULL(X) as a convex polygon in standard form. Let $\mathcal{S}$ be the following algorithm which uses $\mathcal{C}$ to sort a finite set of N distinct positive reals. To avoid trivial cases suppose that $N \geq 3$.

Algorithm $\mathcal{S}$:

Step (1) Input $Y = \{x_1, x_2, \ldots, x_N\}$ where $x_1, x_2, \ldots, x_N$ are distinct postive reals.

Step (2) Form $X = \{(x_1, x_1^2), (x_2, x_2^2), \ldots, (x_N, x_N^2)\}$.

Step (3) Apply $\mathcal{C}$ to X to output the vertices

$$(y_1, y_1^2)\,,\; (y_2, y_2^2)\,,\; \ldots\,,\; (y_N, y_N^2)$$

of the boundary of HULL(X) as a polygon in standard form.

Step (4) Halt and output the sequence $y_1\,,\; y_2\,,\; \ldots\,,\; y_N$.

(a) Use **2**(b) to explain why in **Step (3)**, $\mathcal{C}$ is guaranteed to output N points (not fewer).

(b) Let the running time of $\mathcal{S}$ be $T_{\mathcal{S}}(N)$ and of $\mathcal{C}$ be $T_{\mathcal{C}}(N)$. Explain why

$$T_{\mathcal{S}}(N) \;=\; T_{\mathcal{C}}(N) + O(N)\,.$$

(c) Use the theorem quoted in the preamble to this question and **1**(d) to deduce that

$$T_{\mathcal{C}}(N) \;=\; \Omega(N \log N)\,.$$

(Assume that the relevant limit exists when you apply **1**(d).)

This establishes that the convex hull algorithm in lectures, which has $O(N \log N)$ running time, is optimal.

Figure 2. Last question from the Assignment for MATH2011 in 2003

Students used the solutions to check their answers. Even digesting the solutions provided considerable exam preparation. A majority of the class took the assignment seriously, many breaking off into informal study groups. Figure 3 shows the main exam question relating to the Assignment. Part (a) is an easy lead-in. Part (c) is identical to part of the Assignment, and rewards those who have meticulously read the solutions. Part (b) is similar to the Assignment but sufficiently varied to require careful thought. Part (d) combines all the Assignment, but without its scaffolding, and is only likely to be successfully attempted by someone who has wrestled in depth with the ideas. This exam question was attempted by the majority of students. Six students achieved high distinctions and 12 achieved distinctions, all of whom made progress in the last part; 25 achieved credits and 42 achieved passes, with varying success on the easier parts. Those near the passing threshold typically avoided the question, but demonstrated sufficient competence by making up marks on other questions, despite the disadvantage of ignoring a 15 mark question (out of 75 available marks).

This form of combined assessment clearly succeeded in engaging a significant number of students in learning a new topic independently of the lecturer, collaborating amongst themselves without the slightest fear of behaving unethically or cheating, and being rewarded by achieving high grades. Those students who chose not to expend effort on the assignment could nevertheless pass by demonstrating an appropriate level of competence on other core material. This case study is offered as an example which fulfills the three objectives enunciated in the online document of the Australian Universities Teaching Committee (2002), namely, (i) assessment should guide and encourage effective approaches to learning; (ii) assessment should validly and reliably measure expected learning outcomes; and (iii) assessment and grading should define and protect academic standards.

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B6. Consider nonnegative functions $f(N)$ and $g(N)$ of a natural number N.

(a) Define what is meant by $f(N) = \Omega(g(N))$.

(b) Consider the following conditions, where K is a positive constant:

(i) $\lim_{N\to\infty} \frac{f(N)}{g(N)} = \infty$ (ii) $\lim_{N\to\infty} \frac{f(N)}{g(N)} = 0$ (iii) $\lim_{N\to\infty} \frac{f(N)}{g(N)} = K$

Which of the following (possibly more than one)

(iv) $f(N) = O(g(N))$ (v) $f(N) \neq O(g(N))$

(vi) $f(N) = \Omega(g(N))$ (vii) $f(N) \neq \Omega(g(N))$

are implied by each of (i), (ii), (iii)? (There is no need to supply proofs.)

(c) It is a fact that if $f(N) = O(N)$ and $g(N) + f(N) = \Omega(N \log N)$ then

$$g(N) = \Omega(N \log N).$$

Use limits to verify this, assuming $\lim_{N\to\infty} \frac{g(N)+f(N)}{N \log N}$ exists.

(d) It is a theorem that any sorting algorithm has running time which is $\Omega(N \log N)$. Use this and the result of (c) to sketch a proof that any algorithm which finds the convex hull of a finite set of N points in the plane, represented by vertices of a convex polygon in standard form, also has running time which is $\Omega(N \log N)$.

(Hint: $x_1 < \ldots < x_N$ if and only if $(x_1, x_1^2), \ldots, (x_N, x_N^2)$ are vertices of a convex polygon in standard form.)

[1+3+4+7 = 15 marks]

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Figure 3. Extract from MATH2011 Examination in 2003

References

Australian Universities Teaching Committee (2002) *Core principles of effective assessment.* [Online] Available: http://www.cshe.unimelb.edu.au/assessinglearning/05/index.html [2006, August 15]

Easdown, D. (2003) *A Course in Discrete Mathematics, comprising six instalments.* Sydney: Kopystop, Mountain Street, Broadway.

Easdown, D. (2006a) Teaching mathematics: the gulf between semantics (meaning) and syntax (form). *Proceedings of the 3rd International Conference on the Teaching of Mathematics at the Undergraduate Level.* Istanbul: Turkish Mathematical Society.

Easdown, D. (2006b) Alleviating obstructions to learning. *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Eds. Novotna, Moraova, Kratka and Stehlikova) **1**, 387.

Easdown, D. (2006c) Integrating assessment and feedback to overcome barriers to learning at the passive/active interface in mathematics courses. *Proceedings of the Science Teaching and Learning Research Symposium.* University of Sydney: UniServe Science.

Easdown, D. (2006d) A recent novel approach to assessment and feedback in mathematics courses. *Best Practice in Assessment and Student Feedback Forum.* University of Sydney, [Online] Available: http://www.usyd.edu.au/learning/quality/bp.shtml [2006, August 15]

Rosen, K.H. (2003) *Discrete Mathematics and its Applications.* Fifth Edition. New York: McGraw-Hill.