3×3 Lax pairs for the fourth, fifth and sixth Painlevé equations

N. Joshi,[†] A.V. Kitaev,[†‡] and P.A. Treharne[†]

[†]School of Mathematics and Statistics
University of Sydney
Sydney 2006, Australia,
[‡]Steklov Mathematical Institute
Fontanka 27, St Petersburg, 191023, Russia

Email: N.Joshi@maths.usyd.edu.au, kitaev@pdmi.ras.ru
and P.Treharne@maths.usyd.edu.au

March 2, 2007

Abstract

We obtain 3×3 matrix Lax pairs for systems of ODEs that are solvable in terms of the fourth, fifth and sixth Painlevé equations by considering similarity reductions of the scattering Lax pair for the (2+1)-dimensional three-wave resonant interaction system. These results allow us to construct new 3×3 Lax representations for the fourth and fifth Painlevé equations, together with the previously known 3×3 Lax representation for the sixth Painlevé equation. By comparing these Lax pairs we obtain explicit formulas for the self-similar solutions of the three-wave system in terms of the associated Painlevé equations. Finally, we give a practical application of the 3×3 system associated with the fifth Painlevé equation by using it to derive an Okamoto-type Bäcklund transformation for P_5.

1 Introduction

The Painlevé equations are classical nonlinear second-order ordinary differential equations. They have been the subject of intensive investigation in the last three decades, primarily due to the fact that they appear in connection with a wide range of physical problems, including soliton systems, quantum gravity, string theory and random matrix theory. In this paper we will concentrate on the fourth, fifth and sixth Painlevé equations, the standard forms of which are, respectively

$$\mathrm{P}_4: \quad \frac{d^2y}{dt^2} = \frac{1}{2y}\Big(\frac{dy}{dt}\Big)^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2-\alpha)y + \frac{\beta}{y}, \tag{1.1}$$

$$\mathrm{P}_5: \quad \frac{d^2y}{dt^2} = \Big\{\frac{1}{2y} + \frac{1}{y-1}\Big\}\Big(\frac{dy}{dt}\Big)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{(y-1)^2}{t^2}\Big(\alpha + \frac{\beta}{y}\Big) + \frac{\gamma y}{t} + \frac{\delta y(y+1)}{y-1}, \tag{1.2}$$

$$\begin{aligned}\mathrm{P}_6: \quad \frac{d^2y}{dt^2} &= \frac{1}{2}\Big\{\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\Big\}\Big(\frac{dy}{dt}\Big)^2 - \Big\{\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\Big\}\frac{dy}{dt} \\ &\quad + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\Big\{\alpha + \frac{\beta t}{y} + \frac{\gamma(t-1)}{(y-1)^2} + \frac{\delta t(t-1)}{(y-t)^2}\Big\},\end{aligned} \tag{1.3}$$

where α, β, γ, δ are arbitrary complex parameters.

It is well known that the Painlevé equations P_n govern the isomonodromic deformations of linear 2×2 matrix equations of the form [1]–[3]

$$\frac{dY}{dx} = A^n(x;t)Y, \tag{1.4}$$

where x and t are independent complex variables and $A^n(x;t)$ is a 2×2 matrix that is rational in x. For the equations listed above, namely P_4, P_5 and P_6, Jimbo and Miwa [1]–[3] showed that the matrix $A^n(x;t)$ has the following particular forms:

$$A^4(x;t) = \frac{A_0^4(t)}{x} + A_1^4(t) + xA_2^4(t), \tag{1.5a}$$

$$A^5(x;t) = \frac{A_0^5(t)}{x} + \frac{A_1^5(t)}{x-1} + A_2^5(t), \tag{1.5b}$$

$$A^6(x;t) = \frac{A_0^6(t)}{x} + \frac{A_t^6(t)}{x-t} + \frac{A_1^6(t)}{x-1}. \tag{1.5c}$$

It has recently been observed that the Painlevé equations also govern the isomonodromic deformations of matrix equations of order greater than two. In particular, it was shown in [4], and then later in [5] by different means, that the sixth Painlevé equation P_6 also governs the isomonodromic deformations of a 3×3 system

$$\frac{d\Phi}{d\lambda} = B^n(\lambda;t)\Phi, \tag{1.6}$$

where λ is the new spectral variable and, for $n=6$, $B^6(\lambda;t)$ is a 3×3 matrix that is rational in λ and has the form

$$B^6(\lambda;t) = \frac{B_0^6(t)}{\lambda} + B_1^6(t). \tag{1.7}$$

The question of whether or not the other Painlevé equations P_1–P_5 can also be seen to arise directly as monodromy preserving conditions for linear systems of order three or greater has not been fully answered. We shall show in this paper that, in addition to the generic sixth Painlevé equation, the fourth and the fifth Painlevé equations can also be seen to govern isomonodromic deformations of certain 3×3 systems. In the context of equation (1.6) above, we will see that the matrix $B^n(\lambda;t)$ has the following form for P_4 and P_5, respectively,

$$B^4(\lambda;t) = B_0^4(t) + \lambda B_1^4(t), \tag{1.8}$$

$$B^5(\lambda;t) = \frac{B_0^5(t)}{\lambda} + \frac{B_1^5(t)}{\lambda-1} + B_2^5(t). \tag{1.9}$$

We will also show that there is a simple connection (via the generalized Laplace transform) between these 3×3 representations for P_4, P_5 and P_6 and the 2×2 representations of Jimbo and Miwa given in (1.5).

The particular forms of $B^n(\lambda;t)$ given above in relation to P_4, P_5 and P_6 were obtained by considering certain similarity reductions of the three-dimensional three-wave resonant interaction (3WRI) system. The three-dimensional form of the 3WRI system is given in characteristic coordinates by

$$\frac{\partial u_j}{\partial x_j} = i u_m^* u_n^*, \quad \frac{\partial u_j^*}{\partial x_j} = -i u_m u_n, \quad i^2 = -1, \tag{1.10}$$

where (j, m, n) denotes any permutation of $(1, 2, 3)$, u_j, u_j^* are the complex amplitudes of the wave packets, and star denotes complex conjugation. A complete treatment of the group theoretical properties of the 3WRI system was performed in [7], and all classical similarity reductions for this system were obtained.

Among the various reductions to systems of ODEs, three have been linked to the generic form of the fourth, the fifth and the sixth Painlevé equations, respectively. The reduced systems were integrated in [7] in terms of second order, second degree (SD) type equations which, using results from Bureau *et al* [8], were known to be solvable in terms of the classical Painlevé equations.

In this paper we show how it is possible to integrate the reduced systems without recourse to SD type equations by constructing monodromy Lax pairs. By analysing the particular similarity reductions for P_4, P_5 and P_6 in more detail, and in particular by describing the action of each reduction on the scattering Lax pair for the 3WRI system, we are able to obtain monodromy Lax pairs for the reduced systems and, by extension, for the particular Painlevé equations under investigation also. Comparing these monodromy Lax pairs we derive a direct link between the self-similar solutions to the 3WRI system and the classical Painlevé equations.

The 2×2 systems of Jimbo and Miwa given in (1.5) have come to be identified as model 2×2 equations which can be used to investigate analytic and asymptotic properties of the associated Painlevé equations. An analogous interpretation should also be possible for the 3×3 systems (1.7)–(1.9). Since equations (1.7) and (1.8) each have a singularity structure in the complex λ-plane that is distinct from the 2×2 case of (1.5c) and (1.5a), respectively, the 3×3 systems may in fact provide further insights into the asymptotic properties of the classical Painlevé equations.

In the case of P_5 the singularity structure of the 3×3 equation (1.9) is identical to the 2×2 case of (1.5b), and indeed it is elementary to show that the two representations are related by a simple gauge transformation, see Section 4 below. We note, however, that since (1.9) can also be mapped to the 2×2 matrix system of [1] via the generalized Laplace transform, our 3×3 Lax representation for P_5 provides a new mechanism for generating Bäcklund transformations for solutions of the P_5 equation. In this paper, we will show how to construct the following Bäcklund transformation between solutions of P_5. Assume $y(t)$ and $z(t)$ solve the following system

$$t\frac{dy}{dt} = ty - 2z(y-1)^2 - (y-1)^2\left(\frac{\theta_0 - \theta_1 + \theta_\infty}{2}\right) + (\theta_0 + \theta_1)(y-1), \tag{1.11}$$

$$t\frac{dz}{dt} = yz\left(z + \frac{\theta_0 - \theta_1 + \theta_\infty}{2}\right) - \frac{1}{y}(z + \theta_0)\left(z + \frac{\theta_0 + \theta_1 + \theta_\infty}{2}\right), \tag{1.12}$$

where $\theta_0, \theta_1, \theta_\infty$ are arbitrary complex constants. Eliminating $z(t)$ from this system it follows that $y(t)$ is a solution of P_5 with

$$\alpha = \frac{1}{2}\left(\frac{\theta_0 - \theta_1 + \theta_\infty}{2}\right)^2, \quad \beta = -\frac{1}{2}\left(\frac{\theta_0 - \theta_1 - \theta_\infty}{2}\right)^2, \quad \gamma = 1 - \theta_0 - \theta_1, \tag{1.13}$$

and where t has been rescaled so that $\delta = -1/2$. Define $\hat{y}(t)$ and $\hat{z}(t)$ by

$$\hat{y} = \frac{yz}{z + (\theta_0 + \theta_1 + \theta_\infty)/2}, \tag{1.14}$$

$$\hat{z} = z + \frac{\theta_0 - \theta_1 + \theta_\infty}{2}. \tag{1.15}$$

Then $\hat{y}(t)$ solves (1.2) with

$$\hat{\alpha} = \frac{\theta_1^2}{2}, \quad \hat{\beta} = -\frac{\theta_0^2}{2}, \quad \hat{\gamma} = 1 + \theta_\infty, \quad \hat{\delta} = -\frac{1}{2}. \tag{1.16}$$

We conclude this introduction with some remarks on other Lax representations for the Painlevé equations of order three or greater which have appeared in the literature. Recent work by Noumi and Yamada [9]–[10] has identified 3×3 and 4×4 matrix representations (Lax pairs) for the symmetric forms of the fourth and fifth Painlevé equations, respectively. The symmetry properties of these Lax pairs have been studied in detail by Sen *et al* in [11]–[12], however the connection between these Lax pairs and the standard forms for P_4 and P_5 has not yet been realised. The Lax pairs for the symmetric P_4 and P_5 equations were also derived in [13] by considering periodic reductions of Darboux chains for a Schrödinger problem with quadratic eigenvalue dependence.

The work of Conte *et al* [14] and Kakei and Kikuchi [15] is also relevant since each have obtained (independently) the 3×3 Lax representation for P_6 by considering similarity reductions of the (1+1)-dimensional 3WRI system. While our analysis is directly analogous to the method of [14] and [15], it has the benefit that since it is for the (2+1)-dimensional problem it extends easily to the two-dimensional similiarity reductions identified in [7] and therefore provides a mechanism for generating monodromy Lax pairs for P_4 and P_5.

The paper is organised as follows. In Section 2 we recall the scattering Lax pair for the 3WRI system (1.10). In Section 3 we give a similarity reduction of the 3WRI system to the generic sixth Painlevé equation, P_6. We then describe the action of this similarity reduction on the associated scattering Lax pair and thereby construct a monodromy Lax pair for the reduced system of ODEs. By comparing this Lax pair with the known 3×3 Lax pair for P_6 we obtain explicit formulas for the self-similar solutions of the three-wave system in terms of solutions of P_6. A similar analysis is presented in Sections 4 and 5 for similarity reductions of the 3WRI system to the fifth and the fourth Painlevé equations, respectively, i.e. monodromy Lax pairs are constructed in each case and explicit formulas for the self-similar solutions in terms of solutions of P_5 and P_4 are given. The 3×3 monodromy Lax pairs for P_5 and P_4 that we construct appear to be new. In each section, we also show how to construct the map between the 3×3 systems for P_6, P_5 and P_4 and the 2×2 monodromy Lax pairs of [1] via the generalized Laplace transform. In Section 6 we provide a conclusion and describe the direction of future research. Finally, we give two appendices relating to material presented in this paper. In Appendix A we give a spectral interpretation of the Bäcklund transformations for P_5 in terms of formal monodromy data and characterise the transformation (1.14) given above. Then, for completeness, in Appendix B we give a similarity reduction of the 3WRI system to the third Painlevé equation and identify a connection between the 2×2 mondoromy system of Jimbo and Miwa [1] and a 3×3 system with two regular singularities.

2 Scattering Lax pair for the 3WRI system

System (1.10) admits a scattering Lax pair given by Kaup [20]

$$\begin{aligned} \frac{\partial \psi_j}{\partial x_m} - ik\kappa_m \psi_j &= -iu_n^* \psi_m \\ \frac{\partial \psi_m}{\partial x_j} - ik\kappa_j \psi_m &= iu_n \psi_j \end{aligned} \tag{2.1}$$

where (j, m, n) denotes any permutation of $(1, 2, 3)$, $\psi_j = \psi_j(x_m, k)$ are scalar functions, κ_j are real constants, and $k \in \mathbb{C}$ is the spectral parameter. We note that the inclusion of the spectral parameter k comes from treating system (2.1) as a *scattering* problem and hence making an assumption about the behaviour of the functions ψ_j as $x_m \to \infty$. System (2.1) can be written in matrix form in the following way

$$\begin{aligned} \mathcal{D}_1 \Psi &= i\big(kK_1 + U_1\big)\Psi \\ \mathcal{D}_2 \Psi &= i\big(kK_2 + U_2\big)\Psi \end{aligned} \tag{2.2}$$

where Ψ is a 3×3 matrix-valued function, the matrix operators $\mathcal{D}_1, \mathcal{D}_2$ are given by $\mathcal{D}_1 = \mathrm{diag}[\partial_{x_2}, \partial_{x_3}, \partial_{x_1}]$ and $\mathcal{D}_2 = \mathrm{diag}[\partial_{x_3}, \partial_{x_1}, \partial_{x_2}]$, and the matrices K_1, K_2 and U_1, U_2 are given by

$$K_1 = \begin{pmatrix} \kappa_2 & 0 & 0 \\ 0 & \kappa_3 & 0 \\ 0 & 0 & \kappa_1 \end{pmatrix}, \qquad U_1 = \begin{pmatrix} 0 & -u_3^* & 0 \\ 0 & 0 & -u_1^* \\ -u_2^* & 0 & 0 \end{pmatrix},$$

$$K_2 = \begin{pmatrix} \kappa_3 & 0 & 0 \\ 0 & \kappa_1 & 0 \\ 0 & 0 & \kappa_2 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 0 & 0 & u_2 \\ u_3 & 0 & 0 \\ 0 & u_1 & 0 \end{pmatrix}.$$

We note that, when written in standard cartesian coordinates, the linear system (2.2) is equivalent to the scattering Lax pair identified by Fokas and Ablowitz in [21].

In the following sections we will investigate the particular similarity reductions found in [7] that are linked to P_4, P_5 and P_6, giving an explicit description of the action of each reduction on the scattering Lax pair (2.2).

3 Similarity reduction to the sixth Painlevé equation

The following two-dimensional similarity reduction for system (1.10) was obtained in [6] and [7]

$$v_j(\tau) = (x_m - x_n)^{1 - i\rho_j} u_j, \quad \tau = \frac{x_1 - x_3}{x_2 - x_3}, \tag{3.1}$$

where ρ_1, ρ_2, ρ_3 are real constants such that $\rho_1 + \rho_2 + \rho_3 = 0$. Under this reduction, system (1.10) becomes

$$\begin{aligned} \tau^{1+i\rho_2}(\tau - 1)^{1+i\rho_3} v_1' &= i v_2^* v_3^* \\ \tau^{i\rho_2}(\tau - 1)^{1+i\rho_3} v_2' &= -i v_3^* v_1^* \\ \tau^{1+i\rho_2}(\tau - 1)^{i\rho_3} v_3' &= i v_1^* v_2^*, \end{aligned} \tag{3.2}$$

where prime denotes differentiation with respect to τ.

The above system was integrated directly in [7] in terms of a second order, second degree (SD) equation which, in turn, is solvable in terms of the generic sixth Painlevé equation, P_6. The action of the similarity reduction (3.1) on the scattering Lax pair (2.2) was not discussed in [6], [7]. In the remainder of this section we will show how to find the reduced Lax pair and thereby integrate system (3.2) explicitly in terms of the sixth Painlevé equation, i.e. without recourse to SD type equations.

3.1 A monodromy Lax pair

To compute the reduced Lax pair we introduce the monodromy variable λ in the following way

$$\lambda = (x_2 - x_3)k. \tag{3.3}$$

Writing $\Psi(x_j, k) = R(x_j)\tilde{\Phi}(\tau, \lambda)$, where $R(x_j)$ is given by

$$R(x_1, x_2, x_3) = \mathrm{diag}\left((x_1 - x_3)^{i\theta_{12}}, (x_1 - x_3)^{i\theta_{23}}, (x_1 - x_3)^{i\theta_{31}}\right), \tag{3.4a}$$

and

$$\theta_{12} - \theta_{31} = \rho_1, \quad \theta_{23} - \theta_{12} = \rho_2, \quad \theta_{31} - \theta_{23} = \rho_3, \tag{3.4b}$$

we find that the scattering Lax pair (2.2) becomes

$$\begin{aligned} C_1\tilde{\Phi}_\tau + \lambda D_1\tilde{\Phi}_\lambda &= i\big(\lambda K_1 + V_1\big)\tilde{\Phi} \\ C_2\tilde{\Phi}_\tau + \lambda D_2\tilde{\Phi}_\lambda &= i\big(\lambda K_2 + V_2\big)\tilde{\Phi}, \end{aligned}$$

where the matrices C_j, D_j, K_j, V_j are given by

$$C_1 = \mathrm{diag}\big(-\tau, \tau - 1, 1\big), \qquad C_2 = \mathrm{diag}\big(\tau - 1, 1, -\tau\big),$$

$$D_1 = \mathrm{diag}\big(1, -1, 0\big), \qquad D_2 = \mathrm{diag}\big(-1, 0, 1\big),$$

$$K_1 = \mathrm{diag}\big(\kappa_2, \kappa_3, \kappa_1\big), \qquad K_2 = \mathrm{diag}\big(\kappa_3, \kappa_1, \kappa_2\big),$$

$$V_1 = \begin{pmatrix} 0 & -(1-\tau)^{-1}v_3^* & 0 \\ 0 & \tau^{-1}\theta_{31} & -v_1^* \\ -\tau^{-1}v_2^* & 0 & -\tau^{-1}\theta_{12} \end{pmatrix}, \quad V_2 = \begin{pmatrix} \tau^{-1}\theta_{23} & 0 & \tau^{-1}v_2 \\ (1-\tau)^{-1}v_3 & -\tau^{-1}\theta_{31} & 0 \\ 0 & v_1 & 0 \end{pmatrix}.$$

After rearranging the above system, we find

$$\tilde{\Phi}_\lambda = \Big(Q^{(1)} + \frac{Q^{(0)}}{\lambda}\Big)\tilde{\Phi} \tag{3.5a}$$

$$\tilde{\Phi}_\tau = \Big(\lambda P^{(1)} + P^{(0)}\Big)\tilde{\Phi}, \tag{3.5b}$$

where the matrices $Q^{(1)}, P^{(1)}, Q^{(0)}, P^{(0)}$ are given by

$$Q^{(1)} = i\mathrm{diag}\big(-(\tau-1)\kappa_2 - \tau\kappa_3, (\tau-1)\kappa_1 - \kappa_3, \tau\kappa_1 + \kappa_2\big), \tag{3.6a}$$

$$P^{(1)} = i\mathrm{diag}\big(-\kappa_2 - \kappa_3, \kappa_1, \kappa_1\big), \tag{3.6b}$$

and

$$Q^{(0)} = i\begin{pmatrix} -\theta_{23} & -v_3^* & -v_2 \\ -v_3 & -\theta_{31} & v_1^* \\ -v_2^* & v_1 & -\theta_{12} \end{pmatrix}, \tag{3.6c}$$

$$P^{(0)} = -i\begin{pmatrix} \tau^{-1}\theta_{23} & (\tau-1)^{-1}v_3^* & \tau^{-1}v_2 \\ (\tau-1)^{-1}v_3 & \tau^{-1}\theta_{31} & 0 \\ \tau^{-1}v_2^* & 0 & \tau^{-1}\theta_{12} \end{pmatrix}. \tag{3.6d}$$

In order to integrate the reduced system (3.2) in terms of P_6 we compare the Lax representation (3.5) with the 3×3 Lax representation for P_6 obtained in [4] and [5].

3.2 Solution in terms of the sixth Painlevé equation

We consider the following 3×3 system of matrix equations

$$\Phi_\lambda = \Big(B_1^6 + \frac{B_0^6 - \mathrm{I}}{\lambda}\Big)\Phi \tag{3.7a}$$

$$\Phi_t = \Big(\lambda M_1^6 + M_0^6\Big)\Phi, \tag{3.7b}$$

where the matrices $B_1^6, M_1^6, B_0^6, M_0^6$ are given by

$$B_1^6 = \mathrm{diag}\,\big(t, 1, 0\big), \tag{3.8a}$$

$$M_1^6 = \mathrm{diag}\,\big(1, 0, 0\big), \tag{3.8b}$$

and

$$B_0^6 = \begin{pmatrix} -\theta_2 & \tilde{w}_3 & w_2 \\ w_3 & -\theta_3 & \tilde{w}_1 \\ \tilde{w}_2 & w_1 & -\theta_1 \end{pmatrix}, \tag{3.8c}$$

$$M_0^6 = \begin{pmatrix} -t^{-1}\theta_2 & (t-1)^{-1}\tilde{w}_3 & t^{-1}w_2 \\ (t-1)^{-1}w_3 & -t^{-1}\theta_3 & 0 \\ t^{-1}\tilde{w}_2 & 0 & -t^{-1}\theta_1 \end{pmatrix}, \tag{3.8d}$$

where $\{w_j, \tilde{w}_j\}$ are functions of t and $\theta_1, \theta_2, \theta_3$ are arbitrary constants. If we adopt the parametrisation of [5], then the matrix $B_0^6(t)$ has eigenvalues

$$\mu_1 = \frac{1}{2}\Big(\sum_{j=1}^{3}\theta_j + \theta_\infty\Big), \quad \mu_2 = \frac{1}{2}\Big(\sum_{j=1}^{3}\theta_j - \theta_\infty\Big), \quad \mu_3 = 0,$$

where θ_∞ is an arbitrary constant. It was shown in [5] that P_6 arises as the compatibility condition for this system if the functions $\{w_j, \tilde{w}_j\}$ are given by

$$w_1 = f\left(\frac{\theta_1 + (t-1)y'}{2y} + \frac{1 + \theta_3(t-1) - \theta_1 t + \theta_\infty(y-1) - y}{2t}\right), \tag{3.9a}$$

$$\tilde{w}_1 = f^{-1}\left(-\frac{\theta_3 y - ty'}{2(y-1)} + \frac{\theta_1 t + (\theta_\infty - 1)y}{2(t-1)}\right), \tag{3.9b}$$

$$w_2 = \frac{g}{f}\left(-\frac{\theta_2 y + ty'}{2(y-t)} - \frac{\theta_1 + \theta_\infty y}{2(t-1)} + \frac{y(y-1)}{2(t-1)(y-t)}\right), \tag{3.9c}$$

$$\tilde{w}_2 = \frac{f}{g}\left(\frac{t(\theta_1 + (t-1)y')}{2y} + \frac{1 - \theta_1 - \theta_2(t-1) + \theta_\infty(y-t) - y}{2}\right), \tag{3.9d}$$

$$w_3 = g^{-1}\left(-\frac{\theta_3(y-t) + t(t-1)y'}{2(y-1)} + \frac{\theta_2 t - \theta_\infty(y-t) + y}{2}\right), \tag{3.9e}$$

$$\tilde{w}_3 = g\left(-\frac{(t-1)y' + \theta_2(y-1)}{2(y-t)} + \frac{\theta_3 - \theta_\infty(y-1)}{2t} + \frac{y(y-1)}{2t(y-t)}\right), \tag{3.9f}$$

where $y(t)$ is a solution of P_6 with

$$\alpha = \frac{(\theta_\infty - 1)^2}{2}, \quad \beta = -\frac{\theta_1^2}{2}, \quad \gamma = \frac{\theta_3^2}{2}, \quad \delta = \frac{1 - \theta_2^2}{2},$$

and f, g can be found by quadratures.

In order to solve the reduced 3WRI system (3.2) in terms of P_6 we take $\kappa_1 = 0, \kappa_2 = 0, \kappa_3 = i$ in (2.2), and then compare matrix entries in the monodromy Lax pair (3.5) with $\tau = t$ with those in system (3.7) to obtain the following correspondence:

$$v_1(t) = -iw_1(t), \qquad v_2(t) = iw_2(t), \qquad v_3(t) = iw_3(t), \tag{3.10a}$$
$$v_1^*(t) = -i\tilde{w}_1(t), \qquad v_2^*(t) = i\tilde{w}_2(t), \qquad v_3^*(t) = i\tilde{w}_3(t), \tag{3.10b}$$

and

$$i\rho_1 = \theta_1 - \theta_3, \quad i\rho_2 = \theta_2 - \theta_1, \quad i\rho_3 = \theta_3 - \theta_2. \tag{3.10c}$$

Remark 3.1. We note that the parametrisation that was adopted in [5] to write system (3.7) explicitly in terms of y where $y(t)$ is a solution of P_6 is not unique. Alternate parametrisations have been identified by Boalch [16]–[17] in his investigations of a 3×3 Lax pair with Fuchsian singularities. Since this system can be mapped to the irregular 3×3 Lax pair of [4] and [5] via the generalized Laplace transform, see equation (3.14) below, it follows that these parametrisations are equivalent to system (3.7) up to gauge transformation.

3.3 Reduction to the 2×2 monodromy Lax pair for P_6

In this section we reconstruct the map from the irregular 3×3 Lax representation for P_6 given in (3.7a) to the Fuchsian 2×2 system of Jimbo and Miwa given in (1.5c). We follow the method of [5] using the generalized Laplace transform and appropriate gauge transformations to map equation (3.7a) to another 3×3 system, which is then reducible to the 2×2 system (1.5c). Starting with (3.7a) written in the following form

$$\lambda \Phi_\lambda = \left(\lambda B_1^6(t) + B_0^6(t) - \mathrm{I}\right)\Phi,$$

we introduce the function $\tilde{Y}(x, t)$ via the generalized Laplace transform

$$\Phi(\lambda, t) = \int_C e^{\lambda x} \tilde{Y}(x, t) dx. \tag{3.11}$$

Substituting (3.11) into the above equation, and assuming that the contour C can be chosen to eliminate any remainder terms that arise from integration-by-parts, we find

$$\left(B_1^6(t) - x\mathrm{I}\right)\frac{d\tilde{Y}}{dx} = B_0^6(t)\tilde{Y}. \tag{3.12}$$

We now assume that $B_0^6(t)$ is diagonalisable (which we can do without loss of generality for P_6, see [5]) and, letting G be the diagonalising matrix of $B_0^6(t)$, i.e. $G^{-1}B_0^6(t)G = \mathrm{diag}\,[\mu_1, \mu_2, \mu_3]$, we make the gauge transformation $\tilde{Y} = G^{-1}\hat{Y}$ to find

$$\frac{d\hat{Y}}{dx} = G^{-1}\left(B_1^6(t) - xI\right)^{-1} G\hat{B}_0^6 \hat{Y}, \tag{3.13}$$

where $\hat{B}_0^6 = \mathrm{diag}\,[\mu_1, \mu_2, 0]$. We note that multiplication on $\hat{B}_0^6$ is from the left and so, after an elementary calculation, it follows that this equation can be written as a Fuchsian system

$$\frac{d\hat{Y}}{dx} = \left(\frac{\hat{A}_0^6(t)}{x} + \frac{\hat{A}_t^6(t)}{x-t} + \frac{\hat{A}_1^6(t)}{x-1}\right)\hat{Y}, \tag{3.14}$$

where the 3×3 matrices $\hat{A}_j^6$ all have the form

$$\hat{A}_j^6 = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 0 \end{pmatrix}.$$

Since the third column of each $\hat{A}_j^6$ is zero, the system for $\hat{Y}$ reduces to a system for the first two components

$$\frac{dY}{dx} = \left(\frac{A_0^6(t)}{x} + \frac{A_t^6(t)}{x-t} + \frac{A_1^6(t)}{x-1} \right) Y, \quad Y = \begin{pmatrix} \hat{Y}_1 \\ \hat{Y}_2 \end{pmatrix}, \tag{3.15}$$

and a quadrature for the third component. The eigenvalues of the matrices A_0^6, A_t^6 and A_1^6 are $(\theta_1, 0)$, $(\theta_2, 0)$ and $(\theta_3, 0)$, respectively, see [5]. Equation (3.15) is equivalent (up to gauge transformation) to the 2×2 system of Jimbo and Miwa given in [1], i.e. equation (1.5c).

4 Similarity reduction to the fifth Painlevé equation

In this section we investigate a two-dimensional similarity reduction of the 3WRI system to a system of ODEs that is solvable in terms of the fifth Painlevé equation, P_5. The similarity reduction, which was derived in [7], is given by

$$\begin{aligned} v_1(\tau) &= x_3^{i\rho_1} \exp[ix_2x_3]u_1 \\ v_2(\tau) &= x_3^{i\rho_2} \exp[-ix_3x_1]u_2 \\ v_3(\tau) &= (x_1 - x_2)^{1-i\rho_3}u_3, \quad \tau = (x_1 - x_2)x_3, \end{aligned} \tag{4.1}$$

where ρ_1, ρ_2, ρ_3 are real constants such that $\rho_1 + \rho_2 + \rho_3 = 0$. Under this reduction, system (1.10) becomes

$$\begin{aligned} \tau^{1+i\rho_3} e^{i\tau} v_1' &= i v_2^* v_3^* \\ \tau^{1+i\rho_3} e^{i\tau} v_2' &= -i v_3^* v_1^* \\ \tau^{i\rho_3} e^{i\tau} v_3' &= i v_1^* v_2^*. \end{aligned} \tag{4.2}$$

where prime denotes differentiation with respect to τ. This system was integrated directly in [7] in terms of an SD function and shown to be solvable in terms of the generic fifth Painlevé equation, P_5. Following the approach outlined in the previous section we will use (4.1) to construct a monodromy Lax pair for the reduced system (4.2) and then carry out the explicit integration in terms of P_5.

4.1 A monodromy Lax pair

To compute the reduced Lax pair we introduce the monodromy variable λ in the following way

$$\lambda = (x_1 + x_2)x_3. \tag{4.3}$$

Writing $\tilde{\Psi} = \Psi \exp[ik(\kappa_1 x_1 + \kappa_2 x_2 + \kappa_3 x_3)]$ and then $\tilde{\Psi}(x_j, k) = R(x_j)\tilde{\Phi}(\tau, \lambda)$ where $R(x_j)$ is given by

$$R(x_1, x_2, x_3) = \mathrm{diag}\left(x_3^{-i\theta_{23}} e^{ix_1x_3}, x_3^{-i\theta_{31}} e^{ix_2x_3}, x_3^{-1-i\theta_{12}} \right), \tag{4.4a}$$

and

$$\theta_{12} - \theta_{31} = \rho_1, \quad \theta_{23} - \theta_{12} = \rho_2, \quad \theta_{31} - \theta_{23} = \rho_3, \tag{4.4b}$$

the scattering Lax pair (2.2) becomes

$$\begin{aligned} \tau\tilde{\Phi}_\tau + D_1\tilde{\Phi}_\lambda &= i\Big(-\tfrac{1}{2}(\lambda-\tau)S_2 + V_1\Big)\tilde{\Phi} \\ \tau\tilde{\Phi}_\tau + D_2\tilde{\Phi}_\lambda &= i\Big(-\tfrac{1}{2}(\lambda+\tau)S_1 + V_2\Big)\tilde{\Phi}, \end{aligned}$$

where the matrices D_j, S_j, V_j are given by

$$\begin{aligned} D_1 &= \operatorname{diag}\big(-\tau,\lambda,\tau\big), & D_2 &= \operatorname{diag}\big(\lambda,\tau,-\tau\big), \\ S_1 &= \operatorname{diag}\big(1,0,0\big), & S_2 &= \operatorname{diag}\big(0,1,0\big), \\ V_1 &= \begin{pmatrix} 0 & \tau^{-i\rho_3}e^{-i\tau}v_3^* & 0 \\ 0 & \theta_{31} & -v_1^* \\ -\tau v_2^* & 0 & 0 \end{pmatrix}, & V_2 &= \begin{pmatrix} \theta_{23} & 0 & v_2 \\ \tau^{i\rho_3}e^{i\tau}v_3 & 0 & 0 \\ 0 & -\tau v_1 & 0 \end{pmatrix}. \end{aligned}$$

After rearranging, the above system can be written as

$$\tilde{\Phi}_\lambda = \Big(\frac{\tilde{Q}^{(0)}}{\lambda+\tau} + \frac{\tilde{Q}^{(1)}}{\lambda-\tau} + \tilde{Q}^{(2)}\Big)\tilde{\Phi} \tag{4.5a}$$

$$\tilde{\Phi}_\tau = \Big(\frac{\tilde{P}^{(0)}}{\lambda+\tau} + \frac{\tilde{P}^{(1)}}{\lambda-\tau} + \tilde{P}^{(2)}\Big)\tilde{\Phi} \tag{4.5b}$$

where the matrices $Q^{(0)}, P^{(0)}, Q^{(1)}, P^{(1)}$ and $Q^{(2)}, P^{(2)}$ are given by

$$Q^{(0)} = P^{(0)} = i\begin{pmatrix} \theta_{23} & -\tau^{-i\rho_3}e^{-i\tau}v_3^* & v_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.6a}$$

$$Q^{(1)} = -P^{(1)} = i\begin{pmatrix} 0 & 0 & 0 \\ -\tau^{i\rho_3}e^{i\tau}v_3 & \theta_{31} & -v_1^* \\ 0 & 0 & 0 \end{pmatrix} \tag{4.6b}$$

and

$$Q^{(2)} = -\frac{i}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ v_2^* & -v_1 & 0 \end{pmatrix}, \tag{4.6c}$$

$$P^{(2)} = -\frac{i}{2}\begin{pmatrix} 1 & -2\tau^{-1-i\rho_3}e^{-i\tau}v_3^* & 0 \\ -2\tau^{-1+i\rho_3}e^{i\tau}v_3 & -1 & 0 \\ v_2^* & v_1 & 0 \end{pmatrix}. \tag{4.6d}$$

4.2 Solution in terms of the fifth Painlevé equation

In order to be able to use the monodromy Lax pair (4.5) to integrate the reduced system (4.2) in terms of P_5 we must first construct a 3×3 Lax pair that admits the fifth Painlevé equation directly as the compatibility condition. To do this we make the assumption that the general *form* of the monodromy Lax pair (4.5) gives the correct rational λ-dependence required for a 3×3 monodromy representation of the associated Painlevé equation, P_5. This assumption is based upon the observation that, in the case of P_6, the singularity structure of the 3×3 monodromy Lax pair (3.5) coincides with the singularity structure of the 3×3 system for P_6 identified in [4] and [5]. This leads us to consider the following 3×3 system of matrix equations

$$\Phi_\lambda = \left(\frac{\tilde{B}_0^5}{\lambda + t} + \frac{\tilde{B}_1^5}{\lambda - t} + \tilde{B}_2^5 \right) \Phi \tag{4.7a}$$

$$\Phi_t = \left(\frac{\tilde{M}_0^5}{\lambda + t} + \frac{\tilde{M}_1^5}{\lambda - t} + \tilde{M}_2^5 \right) \Phi, \tag{4.7b}$$

where the matrices $\tilde{B}_j^5, \tilde{M}_j^5$ are given by

$$\tilde{B}_0^5 = \tilde{M}_0^5 = \begin{pmatrix} \tilde{m} & \tilde{w}_3 & w_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.8a}$$

$$\tilde{B}_1^5 = -\tilde{M}_1^5 = \begin{pmatrix} 0 & 0 & 0 \\ w_3 & m & \tilde{w}_1 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.8b}$$

and

$$\tilde{B}_2^5 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \tilde{w}_2 & w_1 & 0 \end{pmatrix} \tag{4.8c}$$

$$\tilde{M}_2^5 = \frac{1}{2} \begin{pmatrix} 1 & -2t^{-1}\tilde{w}_3 & 0 \\ -2t^{-1}w_3 & -1 & 0 \\ \tilde{w}_2 & -w_1 & 0 \end{pmatrix}, \tag{4.8d}$$

and $\{w_j, \tilde{w}_j\}$ are all functions of t. Compatibility of equations (4.7a) and (4.7b) gives the following system of equations

$$m' = 0, \quad \tilde{m}' = 0, \tag{4.9}$$

and

$$\begin{aligned} tw_1' &= \tilde{w}_2\tilde{w}_3, & t\tilde{w}_1' &= -w_2w_3, \\ tw_2' &= -\tilde{w}_1\tilde{w}_3, & t\tilde{w}_2' &= w_1w_3, \\ tw_3' &= -[t - (m - \tilde{m})]w_3 - t\tilde{w}_1\tilde{w}_2, & t\tilde{w}_3' &= [t - (m - \tilde{m})]\tilde{w}_3 + tw_1w_2, \end{aligned} \tag{4.10}$$

This system admits first integrals

$$\begin{aligned} &m = c_1, \quad \tilde{m} = c_2, \quad w_1\tilde{w}_1 + w_2\tilde{w}_2 = c_3 \\ &w_1w_2w_3 + \tilde{w}_1\tilde{w}_2\tilde{w}_3 + w_3\tilde{w}_3 - \tilde{m}w_1\tilde{w}_1 - mw_2\tilde{w}_2 = c_4, \end{aligned} \tag{4.11}$$

where c_j are constants. Without loss of generality, we choose

$$\begin{aligned} &c_1 = -\big(\theta_0 + \theta_1 + \theta_\infty\big)/2, \quad c_2 = -\big(\theta_0 + \theta_1 - \theta_\infty\big)/2 \\ &c_3 = \theta_0, \quad c_4 = (\theta_0 + \theta_1 - \theta_\infty)(\theta_0 + \theta_1 + \theta_\infty)/4, \end{aligned} \tag{4.12}$$

where $\theta_0, \theta_1, \theta_\infty$ are arbitrary constants. If we define the function $y(t)$ as follows

$$y(t) = \nu \frac{(z + \theta_0)}{\big(z + (\theta_0 - \theta_1 + \theta_\infty)/2\big)}, \tag{4.13a}$$

where

$$z+\theta_0 = w_2\tilde{w}_2, \quad \nu = \frac{\tilde{w}_3 + w_1 w_2}{w_1 w_2}, \tag{4.13b}$$

then, from system (4.10), we have

$$ty' = ty - 2z(y-1)^2 + (y-1)\left(\frac{\theta_0+\theta_1-\theta_\infty}{2}y + \frac{\theta_0+\theta_1+\theta_\infty}{2}\right), \tag{4.14a}$$

$$tz' = y\left(z-\frac{\theta_0}{2}\right)\left(z-\frac{\theta_1-\theta_\infty}{2}\right) - \frac{1}{y}\left(z+\frac{\theta_0}{2}\right)\left(z+\frac{\theta_1+\theta_\infty}{2}\right), \tag{4.14b}$$

$$t\nu' = t\nu + \theta_\infty(\nu-1) - (2z+\theta_0)(\nu-1)^2, \tag{4.14c}$$

and it follows that $y(t)$ satisfies the classical P_5 equation with δ scaled to $\delta = -1/2$ and

$$\alpha = \frac{1}{2}\left(\frac{\theta_0-\theta_1+\theta_\infty}{2}\right)^2, \quad \beta = -\frac{1}{2}\left(\frac{\theta_0-\theta_1-\theta_\infty}{2}\right)^2, \quad \gamma = 1-\theta_0-\theta_1. \tag{4.15}$$

We note that, with the change of variables $\lambda \mapsto t(2\lambda-1)$, equation (4.7a) gives the form for $B(\lambda;t)$ specified in equation (1.9).

Using the parametrisation for $y(t)$ given in (4.13) we obtain the following expressions for the functions $\{w_j(t), \tilde{w}_j(t)\}$:

$$w_1 = -gz^{1/2}(z+\theta_0)^{1/2}, \tag{4.16a}$$

$$\tilde{w}_1 = \frac{1}{g}\frac{z^{1/2}}{(z+\theta_0)^{1/2}}, \tag{4.16b}$$

$$w_2 = \frac{1}{f}\frac{(z+\theta_0)^{1/2}}{z^{1/2}}, \tag{4.16c}$$

$$\tilde{w}_2 = fz^{1/2}(z+\theta_0)^{1/2}, \tag{4.16d}$$

$$w_3 = \frac{f}{g}\left(\frac{1}{y}\Big(z+\frac{\theta_0+\theta_1+\theta_\infty}{2}\Big) - z\right), \tag{4.16e}$$

$$\tilde{w}_3 = \frac{g}{f}\left(-y\Big(z+\frac{\theta_0-\theta_1+\theta_\infty}{2}\Big) + z + \theta_0\right), \tag{4.16f}$$

where, from system (4.10), the functions $f(t), g(t)$ satisfy the following equations

$$t(\log f)' = -z + \frac{yz}{(z+\theta_0)}\left(z+\frac{\theta_0-\theta_1+\theta_\infty}{2}\right) + \frac{\theta_0 tz'}{2z^{3/2}(z+\theta_0)^{1/2}}, \tag{4.17a}$$

$$t(\log g)' = z+\theta_0 + \frac{(z+\theta_0)}{yz}\left(z+\frac{\theta_0+\theta_1+\theta_\infty}{2}\right) + \frac{\theta_0 tz'}{2z^{1/2}(z+\theta_0)^{3/2}}. \tag{4.17b}$$

In order to solve the reduced 3WRI system (4.2) in terms of P_5 we compare matrix entries in the monodromy system (4.5) with $\tau = t$ with those in (4.7) to get the following correspondence:

$$v_1(t) = -iw_1(t), \qquad v_2(t) = -iw_3(t), \qquad v_3(t) = it^{-i\rho_3}e^{-it}w_3(t), \tag{4.18a}$$

$$v_1^*(t) = i\tilde{w}_1(t), \qquad v_2^*(t) = i\tilde{w}_2(t), \qquad v_3^*(t) = it^{i\rho_3}e^{it}\tilde{w}_3(t), \tag{4.18b}$$

and

$$i\theta_{23} = -\left(\frac{\theta_0+\theta_1-\theta_\infty}{2}+1\right), \quad i\theta_{31} = -\left(\frac{\theta_0+\theta_1+\theta_\infty}{2}+1\right). \tag{4.18c}$$

Note that if we assume $\theta_{12} = 0$ in equation (4.4b) then ρ_1, ρ_2, ρ_3 can be expressed in terms of $\theta_0, \theta_1, \theta_\infty$ as

$$i\rho_1 = \left(\frac{\theta_0 + \theta_1 + \theta_\infty}{2} + 1\right), \quad i\rho_2 = -\left(\frac{\theta_0 + \theta_1 - \theta_\infty}{2} + 1\right), \quad i\rho_3 = -\theta_\infty. \tag{4.19}$$

4.3 Reductions to the 2×2 monodromy Lax pair for $\mathbf{P}_5$

In this section we show how to reduce the 3×3 monodromy Lax pair for P_5, system (4.7a), to the 2×2 linear system of [1] given in (1.5b). The first approach we take is to use the generalized Laplace transform (3.11). We start by writing (4.7a) in the form

$$(\lambda \hat{B}_2^5 + \hat{B}_1^5)\Phi_\lambda = \big((\lambda \hat{B}_2^5 + \hat{B}_1^5) + \hat{B}_0^5\big)\Phi, \tag{4.20}$$

where

$$\hat{B}_2^5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{B}_1^5 = \frac{1}{2}\begin{pmatrix} t & 0 & 0 \\ 0 & -t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \hat{B}_0^5 = \begin{pmatrix} \tilde{m} & \tilde{w}_3 & w_2 \\ w_3 & m & \tilde{w}_1 \\ \frac{1}{2}\tilde{w}_2 & \frac{1}{2}w_1 & -\frac{1}{2} \end{pmatrix}. \tag{4.21}$$

Substituting the formula for $\Phi(\lambda, t)$ from equation (3.11) into equation (4.20), and assuming that the contour C can be suitably chosen to eliminate any remainder terms that arise from integration-by-parts, we find

$$(x-1)\hat{B}_2^5 \frac{d\tilde{Y}}{dx} = \big((x-1)\hat{B}_1^5 - (\hat{B}_2^5 + \hat{B}_0^5)\big)\tilde{Y}. \tag{4.22}$$

Since the diagonal matrix $\hat{B}_2^5$ has a zero in the (33) position, it follows that the third row of this expression gives a relationship between the elements of $\tilde{Y}$:

$$x\tilde{Y}_3 = \tilde{w}_2\tilde{Y}_1 + w_1\tilde{Y}_2. \tag{4.23}$$

Using this relation to eliminate $\tilde{Y}_3$ from (4.22) we obtain the following 2×2 system:

$$\frac{dY}{dx} = \left[\frac{t}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{x}\begin{pmatrix} A_0 + 1 & B_0 \\ C_0 & D_0 + 1 \end{pmatrix} + \frac{1}{x-1}\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}\right] Y, \quad Y = \begin{pmatrix} \tilde{Y}_1 \\ \tilde{Y}_2 \end{pmatrix}, \tag{4.24}$$

where

$$\begin{aligned} &A_0 = w_2\tilde{w}_2, \quad B_0 = w_1 w_2, \quad C_0 = \tilde{w}_1\tilde{w}_2, \quad D_0 = w_1\tilde{w}_1, \\ &A_1 = -\big(w_2\tilde{w}_2 + \tilde{m}\big), \quad B_1 = -\big(\tilde{w}_3 + w_1 w_2\big), \\ &C_1 = -\big(w_3 + \tilde{w}_1\tilde{w}_2\big), \quad D_1 = -\big(w_1\tilde{w}_1 + m\big). \end{aligned} \tag{4.25}$$

System (4.24) is equivalent (up to a gauge transformation) to the 2×2 system of Jimbo–Miwa given in [1], i.e. equation (1.5b). Adopting the parametrisation used by Jimbo–Miwa, it follows that $y(t)$ given by (4.13) is a solution of the fifth Painlevé equation.

There is also an alternate reduction of (4.7) to the Jimbo–Miwa system (1.5b) which makes use of suitable gauge transformations rather than the generalized Laplace transform. Starting with equation (4.20) we note that choosing the constant c_4 as in (4.12) implies that the matrix $\hat{B}_0^5$ has zero determinant. Indeed, from the above expression for $\hat{B}_0^5$ we have

$$\det(\hat{B}_0^5) = w_1 w_2 w_3 + \tilde{w}_1\tilde{w}_2\tilde{w}_3 + w_3\tilde{w}_3 - \tilde{m}w_1\tilde{w}_1 - m w_2\tilde{w}_2 - m\tilde{m},$$

which is zero from (4.11) and (4.12). It follows that $\hat{B}_0^5$ has eigenvalues $(\mu_1, \mu_2, 0)$, and so there exists a matrix G such that

$$G^{-1}\hat{B}_0^5 G = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If we now make the gauge transformation $\Phi = G\tilde{\mathcal{Y}}$ then equation (4.20) becomes

$$\frac{d\tilde{\mathcal{Y}}}{d\lambda} = \big(\mathrm{I} + G^{-1}(\lambda\hat{B}_2^5 + \hat{B}_1^5)^{-1}G\hat{\hat{B}}_0^5\big)\tilde{\mathcal{Y}},$$

where $\hat{\hat{B}}_0^5 = \mathrm{diag}\,[\mu_1, \mu_2, 0]$. Since multiplication on $\hat{\hat{B}}_0^5$ is from the left if follows that this can be rewritten further as

$$\frac{d\tilde{\mathcal{Y}}}{d\lambda} = \left[\mathrm{I} + \tilde{A}_2^5 + \frac{\tilde{A}_1^5}{\lambda - t} + \frac{\tilde{A}_0^5}{\lambda + t}\right]\tilde{\mathcal{Y}},$$

where

$$\tilde{A}_2 = \frac{1}{\det(G)}\begin{pmatrix} (G_{12}G_{23} - G_{13}G_{22})G_{31}\mu_1 & (G_{12}G_{23} - G_{13}G_{22})G_{32}\mu_2 & 0 \\ (G_{13}G_{21} - G_{11}G_{23})G_{31}\mu_1 & (G_{13}G_{21} - G_{11}G_{23})G_{32}\mu_2 & 0 \\ (G_{11}G_{22} - G_{12}G_{21})G_{31}\mu_1 & (G_{11}G_{22} - G_{12}G_{21})G_{32}\mu_2 & 0 \end{pmatrix},$$
$$\tilde{A}_1 = \frac{1}{\det(G)}\begin{pmatrix} (G_{13}G_{32} - G_{12}G_{33})G_{21}\mu_1 & (G_{13}G_{32} - G_{12}G_{33})G_{22}\mu_2 & 0 \\ (G_{11}G_{33} - G_{13}G_{31})G_{21}\mu_1 & (G_{11}G_{33} - G_{13}G_{31})G_{22}\mu_2 & 0 \\ (G_{12}G_{31} - G_{11}G_{32})G_{21}\mu_1 & (G_{12}G_{31} - G_{11}G_{32})G_{22}\mu_2 & 0 \end{pmatrix},$$
$$\tilde{A}_0 = \frac{1}{\det(G)}\begin{pmatrix} (G_{22}G_{33} - G_{23}G_{32})G_{11}\mu_1 & (G_{22}G_{33} - G_{23}G_{32})G_{12}\mu_2 & 0 \\ (G_{23}G_{31} - G_{21}G_{33})G_{11}\mu_1 & (G_{23}G_{31} - G_{21}G_{33})G_{12}\mu_2 & 0 \\ (G_{21}G_{32} - G_{22}G_{31})G_{11}\mu_1 & (G_{21}G_{32} - G_{22}G_{31})G_{12}\mu_2 & 0 \end{pmatrix},$$

and G_{ij} are the entries of G. The zeros in the (13) and (23) entries of these coefficient matrices imply that this system is equivalent to a 2×2 system for the first two components of $\tilde{\mathcal{Y}}$,

$$\frac{d\hat{\mathcal{Y}}}{d\lambda} = \Big(\hat{A}_2^5 + \frac{\hat{A}_1^5}{\lambda - t} + \frac{\hat{A}_0^5}{\lambda + t}\Big)\hat{\mathcal{Y}}, \quad \hat{\mathcal{Y}} = \begin{pmatrix} \tilde{\mathcal{Y}}_1 \\ \tilde{\mathcal{Y}}_2 \end{pmatrix},$$

and a quadrature for the third component. The eigenvalues of the matrices $\hat{A}_2^5$, $\hat{A}_1^5$ and $\hat{A}_0^5$ are $(1/2, 0)$, $(-(\theta_0 + \theta_1 + \theta_\infty)/2, 0)$ and $(-(\theta_0 + \theta_1 - \theta_\infty)/2, 0)$, respectively.

We now make a gauge transformation $\hat{\mathcal{Y}} = H\mathcal{Y}$, where H is the diagonalising matrix for $\hat{A}_2^5$, and introduce a change of variables $x = (\lambda + t)/2t$, to get

$$\frac{d\mathcal{Y}}{dx} = \left[\frac{t}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{x}\begin{pmatrix} \hat{A}_0 & \hat{B}_0 \\ \hat{C}_0 & \hat{D}_0 \end{pmatrix} + \frac{1}{x-1}\begin{pmatrix} \hat{A}_1 & \hat{B}_1 \\ \hat{C}_1 & \hat{D}_1 \end{pmatrix}\right]\mathcal{Y}, \tag{4.26}$$

where

$$\begin{aligned} &\hat{A}_0 + \hat{D}_0 = -\frac{\theta_0 + \theta_1 - \theta_\infty}{2}, \quad \hat{A}_0\hat{D}_0 - \hat{B}_0\hat{C}_0 = 0, \\ &\hat{A}_1 + \hat{D}_1 = -\frac{\theta_0 + \theta_1 + \theta_\infty}{2}, \quad \hat{A}_1\hat{D}_1 - \hat{B}_1\hat{C}_1 = 0, \\ &\hat{A}_0 + \hat{A}_1 = -\theta_1. \end{aligned} \tag{4.27}$$

System (4.26) is equivalent (up to a gauge transformation) to the 2×2 system of Jimbo–Miwa given in [1], i.e. equation (1.5b). Adopting the parametrisation used by Jimbo–Miwa, it follows that $\hat{y}(t)$ given by

$$\begin{aligned}\hat{y}(t) &= \frac{(\hat{A}_0-\tilde{m})\hat{B}_1}{\hat{A}_1\hat{B}_0}\\ &= \frac{\tilde{w}_1[mw_2-\tilde{w}_1\tilde{w}_3]}{w_2[\tilde{m}\tilde{w}_1-w_2w_3]}\left(1+\frac{(m+\theta_0)\tilde{w}_1}{(m+\theta_0)\tilde{w}_1+w_2w_3}\left(1-\frac{\tilde{w}_1[mw_2-\tilde{w}_1\tilde{w}_3]}{w_2[\tilde{m}\tilde{w}_1-w_2w_3]}\right)\right),\end{aligned} \tag{4.28}$$

is a solution of the fifth Painlevé equation with $\hat{\delta}$ scaled to $\hat{\delta}=-1/2$ and

$$\hat{\alpha}=\frac{1}{2}\left(\frac{\hat{\theta}_0-\hat{\theta}_1+\hat{\theta}_\infty}{2}\right)^2,\quad \hat{\beta}=-\frac{1}{2}\left(\frac{\hat{\theta}_0-\hat{\theta}_1-\hat{\theta}_\infty}{2}\right)^2,\quad \hat{\gamma}=1-\hat{\theta}_0-\hat{\theta}_1,$$

where $\hat{\theta}_0,\hat{\theta}_1,\hat{\theta}_\infty$ are given by

$$\hat{\theta}_0=\frac{\theta_0+\theta_1-\theta_\infty}{2},\quad \hat{\theta}_1=-\frac{\theta_0+\theta_1+\theta_\infty}{2},\quad \hat{\theta}_\infty=\theta_1-\theta_0.$$

Simplifying the above expressions for $\hat{\alpha},\hat{\beta},\hat{\gamma}$ and substituting the expressions for $\{w_j,\tilde{w}_j\}$ in (4.16) into (4.28) we obtain the Bäcklund transformation for P_5 given in (1.14).

5 Similarity reduction to the fourth Painlevé equation

The following similarity reduction was derived in [7] and reduces system (1.10) to a system of ODEs that are solvable in terms of the fourth Painlevé equation. The similarity reduction is given by

$$v_j(\tau)=e^{-i\theta_j}u_j, \tag{5.1}$$

where

$$\begin{aligned}&\theta_1=-\rho_1x_3+\tfrac{1}{4}x_3^2+x_2x_3,\quad \theta_3=\rho_3(x_1+x_2)+\tfrac{1}{2}(x_1+x_2)^2,\\ &\theta_2=-\rho_2x_3+\tfrac{1}{4}x_3^2+x_3x_1,\quad \tau=x_1+x_2+x_3,\end{aligned} \tag{5.2}$$

and ρ_1,ρ_2,ρ_3 are real constants such that $\rho_1+\rho_2+\rho_3=0$. Under this reduction, system (1.10) becomes

$$\begin{aligned}e^{i\theta}v_1'&=iv_2^*v_3^*,\\ e^{i\theta}v_2'&=iv_3^*v_1^*,\\ e^{i\theta}v_3'&=iv_1^*v_2^*,\quad \theta=\rho_3\tau+\tfrac{1}{2}\tau^2,\end{aligned} \tag{5.3}$$

where prime denotes differentiation with respect to τ. This system was integrated directly in [7] in terms of SD functions and shown to be solvable in terms of the generic fourth Painlevé equation, P_4. Following the approach outlined in the previous sections, we will use (5.1) to construct a monodromy Lax pair for the reduced system (5.3) and then carry out the explicit integration in terms of P_4.

5.1 A monodromy Lax pair

To compute the reduced Lax pair we introduce the monodromy variable λ in the following way

$$\lambda = x_1 - x_2. \tag{5.4}$$

Writing $\tilde{\Psi} = \Psi \exp[ik(\kappa_1 x_1 + \kappa_2 x_2 + \kappa_3 x_3)]$ and then $\tilde{\Psi}(x_j, k) = R(x_j)\tilde{\Phi}(\tau, \lambda)$ where $R(x_j)$ is given by

$$R(x_1, x_2, x_3) = \mathrm{diag}\,\Big(e^{i\theta_2}, e^{-i\theta_1}, 1\Big),$$

the scattering Lax pair (2.2) becomes

$$\begin{aligned}
\tilde{\Phi}_\tau + D_1\tilde{\Phi}_\lambda &= i\big(-\tfrac{1}{2}(\lambda - \tau)S_2 + V_1\big)\tilde{\Phi}\\
\tilde{\Phi}_\tau + D_2\tilde{\Phi}_\lambda &= i\big(-\tfrac{1}{2}(\lambda + \tau)S_1 + V_2\big)\tilde{\Phi},
\end{aligned}$$

where the matrices D_j, S_j, V_j are given by

$$\begin{aligned}
D_1 &= \mathrm{diag}\,\big(-1, 0, 1\big), & D_2 &= \mathrm{diag}\,\big(0, 1, -1\big),\\
S_1 &= \mathrm{diag}\,\big(1, 0, 0\big), & S_2 &= \mathrm{diag}\,\big(0, 1, 0\big),\\
V_1 &= \begin{pmatrix} 0 & -e^{-i\theta}v_3^* & 0\\ 0 & -\rho_1 & -v_1^*\\ -v_2^* & 0 & 0\end{pmatrix}, & V_2 &= \begin{pmatrix} \rho_2 & 0 & v_2\\ e^{i\theta}v_3 & 0 & 0\\ 0 & v_1 & 0\end{pmatrix}.
\end{aligned}$$

After rearranging, the above system can be written in the form

$$\tilde{\Phi}_\lambda = \Big(\lambda Q^{(1)} + Q^{(0)}\Big)\tilde{\Phi} \tag{5.5a}$$

$$\tilde{\Phi}_\tau = \Big(\lambda P^{(1)} + P^{(0)}\Big)\tilde{\Phi}, \tag{5.5b}$$

where the matrices $Q^{(1)}, P^{(1)}, Q^{(0)}, P^{(0)}$ are given by

$$\begin{aligned}
Q^{(1)} &= -\frac{i}{2}\mathrm{diag}\,\big(1, -1, 0\big),\\
P^{(1)} &= -\frac{i}{2}\mathrm{diag}\,\big(1, 1, 0\big),
\end{aligned}$$

and

$$\begin{aligned}
Q^{(0)} &= i\begin{pmatrix} \rho_1 - \frac{1}{2}\tau & e^{-i\theta}v_3^* & v_2\\ e^{i\theta}v_3 & \rho_2 - \frac{1}{2}\tau & v_1^*\\ -\frac{1}{2}v_2^* & -\frac{1}{2}v_1 & 0\end{pmatrix},\\
P^{(0)} &= i\begin{pmatrix} \rho_1 - \frac{1}{2}\tau & 0 & v_2\\ 0 & -\rho_2 + \frac{1}{2}\tau & -v_1^*\\ -\frac{1}{2}v_2^* & \frac{1}{2}v_1 & 0\end{pmatrix}.
\end{aligned}$$

5.2 Solution in terms of the fourth Painlevé equation

We consider the following 3×3 system of matrix equations

$$\Phi_\lambda = \Big(\lambda B_1^4 + B_0^4\Big)\Phi \tag{5.7a}$$

$$\Phi_t = \Big(\lambda M_1^4 + M_0^4\Big)\Phi, \tag{5.7b}$$

where the matrices $B_1^4, M_1^4, B_0^4, M_0^4$ are given by

$$B_1^4 = -\mathrm{diag}\,\big(1, -1, 0\big), \tag{5.8a}$$
$$M_1^4 = -\mathrm{diag}\,\big(1, 1, 0\big), \tag{5.8b}$$

and

$$B_0^4 = \begin{pmatrix} -t & \tilde{w}_3 & w_2 \\ w_3 & -t & \tilde{w}_1 \\ \tilde{w}_2 & w_1 & 0 \end{pmatrix}, \tag{5.8c}$$
$$M_0^4 = \begin{pmatrix} -t & 0 & w_2 \\ 0 & t & -\tilde{w}_1 \\ \tilde{w}_2 & -w_1 & 0 \end{pmatrix}, \tag{5.8d}$$

and $\{w_j, \tilde{w}_j\}$ are all functions of t. Compatibility of equations (5.7a) and (5.7b) gives the following system of equations

$$\begin{aligned} w_1' &= -\tilde{w}_2\tilde{w}_3, & \tilde{w}_1' &= w_2 w_3, \\ w_2' &= -\tilde{w}_1\tilde{w}_3, & \tilde{w}_2' &= w_1 w_3, \\ w_3' &= -2tw_3 - 2\tilde{w}_1\tilde{w}_2, & \tilde{w}_3' &= 2t\tilde{w}_3 + 2w_1w_2. \end{aligned} \tag{5.9}$$

This system admits first integrals

$$\begin{aligned} w_1\tilde{w}_1 - w_2\tilde{w}_2 &= 2\theta_0, \\ w_1\tilde{w}_1 + w_2\tilde{w}_2 - w_3\tilde{w}_3 &= -2\theta_1, \end{aligned} \tag{5.10}$$

with θ_0 and θ_1 constants. Elementary computation now shows that the functions y and $\tilde{y}$, given by

$$y(t) = -\frac{2w_1w_2}{\tilde{w}_3}, \quad \tilde{y}(t) = -\frac{2\tilde{w}_1\tilde{w}_2}{w_3}, \tag{5.11}$$

satisfy the classical P_4 equation (1.1),

$$\begin{aligned} \frac{d^2y}{dt^2} &= \frac{1}{2y}\left(\frac{dy}{dt}\right)^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2-\alpha)y + \frac{\beta}{y}, \\ \frac{d^2\tilde{y}}{dt^2} &= \frac{1}{2\tilde{y}}\left(\frac{d\tilde{y}}{dt}\right)^2 + \frac{3}{2}\tilde{y}^3 + 4t\tilde{y}^2 + 2(t^2-\tilde{\alpha})\tilde{y} + \frac{\tilde{\beta}}{\tilde{y}}, \end{aligned}$$

with $\alpha = -2\theta_1 + 1$ and $\beta = -8\theta_0^2$ for the y equation, and $\tilde{\alpha} = -2\theta_1 - 1$ and $\tilde{\beta} = -8\theta_0^2$ for the $\tilde{y}$ equation.

Using the parametrisation for $y(t), \tilde{y}(t)$ given by equation (5.11) we obtain the following expressions for the functions $\{w_j(t), \tilde{w}_j(t)\}$:

$$(\log w_1)' = -\frac{2\tilde{y}\big(\tilde{y}' - 2t\tilde{y} - \tilde{y}^2 + 4\theta_1\big)}{\big(\tilde{y}' - 2t\tilde{y} - \tilde{y}^2 + 4\theta_0\big)}, \quad (\log \tilde{w}_1)' = \frac{2y\big(y' + 2ty + y^2 + 4\theta_1\big)}{\big(y' + 2ty + y^2 + 4\theta_0\big)}, \tag{5.12a}$$
$$(\log w_2)' = -\frac{\tilde{y}\big(\tilde{y}' - 2t\tilde{y} - \tilde{y}^2 + 4\theta_1\big)}{\big(\tilde{y}' - 2t\tilde{y} - \tilde{y}^2 - 4\theta_0\big)}, \quad (\log \tilde{w}_2)' = \frac{y\big(y' + 2ty + y^2 + 4\theta_1\big)}{\big(y' + 2ty + y^2 - 4\theta_0\big)}, \tag{5.12b}$$
$$(\log w_3)' = -(\tilde{y} + 2t), \qquad (\log \tilde{w}_3)' = y + 2t. \tag{5.12c}$$

In order to solve the reduced 3WRI system (5.3) in terms of P_4 we compare the monodromy systems (5.5) and (5.7) to get the following correspondence:

$$v_1(t) = 2iw_1, \qquad v_2(t) = -iw_2, \qquad v_3(t) = -ie^{-i(2t^2-\rho_3^2/2)}w_3, \tag{5.13a}$$

$$v_1^*(t) = -i\tilde{w}_1, \qquad v_2^*(t) = 2i\tilde{w}_2, \qquad v_3^*(t) = -ie^{i(2t^2-\rho_3^2/2)}\tilde{w}_3, \tag{5.13b}$$

where we have chosen $\rho_1 = \rho_2 = -\rho_3/2$ and made the change of variables $t = (\tau + \rho_3)/2$.

5.3 Reduction to the 2×2 monodromy Lax pair for P_4

In this section we use the generalized Laplace transform (3.11) to construct the map between the 3×3 Lax representation for P_4 given in (5.7) and the 2×2 system of Jimbo and Miwa given in (1.5a). Substituting the formula for $\Phi(\lambda, t)$ from equation (3.11) into equation (5.7a), and assuming that the contour C is suitably chosen to eliminate any remainder terms that arise from integration-by-parts, we find

$$B_1^4 \frac{d\tilde{Y}}{dx} = \big(-x\mathrm{I} + B_0^4 \big)\tilde{Y}, \tag{5.14}$$

where the matrices B_1^4, B_0^4 are given in (5.8a) and (5.8c), respectively. We note that, because the diagonal matrix B_1^4 has a zero in the (33) entry, the third row of this equation gives the following relationship between the components of $\tilde{Y}$

$$x\tilde{Y}_3 = \tilde{w}_2\tilde{Y}_1 + w_1\tilde{Y}_2.$$

Using this expression to eliminate $\tilde{Y}_3$ from (5.14) we are able to reduce equation (5.14) to the following 2×2 system

$$\frac{d\hat{Y}}{dx} = \left(xA_2^4(t) + A_1^4(t) + \frac{A_0^4(t)}{x} \right)\hat{Y}, \quad \hat{Y} = \begin{pmatrix} \tilde{Y}_1 \\ \tilde{Y}_2 \end{pmatrix}, \tag{5.15}$$

where

$$A_0^4(t) = \begin{pmatrix} -w_2\tilde{w}_2 & -w_2w_1 \\ \tilde{w}_2\tilde{w}_1 & w_1\tilde{w}_1 \end{pmatrix}, \quad A_1^4(t) = \begin{pmatrix} t & -\tilde{w}_3 \\ w_3 & -t \end{pmatrix}, \quad A_2^4(t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.16}$$

We note that this system is related to the Jimbo-Miwa system of [1] by a simple gauge transformation.

6 Conclusion

In this paper we have constructed linear 3×3 matrix representations (Lax pairs) for the generic fourth and fifth Painlevé equations, respectively, see equations (5.7) and (4.7). The 3×3 systems are directly analogous to the 3×3 Lax pair associated with the generic sixth Painlevé equation, P_6, which was first derived in [4], and then later in [5] and [14], [15] by different means. Indeed, as we have shown here, all three Lax representations arise from classical similarity reductions of the full three-dimensional three-wave resonant interaction system.

The existence of the 3×3 systems for P_4 and P_5 raises the question of alternate monodromy Lax pair representations for the Painlevé equations which are distinct from the 2×2 cases identified in [1] by Jimbo and Miwa. In the case of P_6, the connection between the 2×2 Lax pair of [1] and the 3×3 Lax pair of [4] has been established by two different methods (by factorization of a residue [4], and by Laplace transform in λ-space [5]). We have shown that a similar connection can be established between our monodromy Lax representations for P_4 and P_5 and the 2×2 systems of Jimbo and Miwa [1] via the Laplace transform in λ-space.

As far as we are aware, the isomonodromy problems for the λ-part of equations (4.7) and (5.7) have not been studied in detail. The fact that our 3×3 system for P_4 possesses an irregular singularity at infinity and no other singularities in the complex λ-plane establishes a clear distinction between this system and the isomonodromy problem of Jimbo and Miwa in [1], i.e. equation (1.5a). It will be interesting to determine the monodromy data for equation (5.7) and to investigate whether or not it can be used to determine all Bäcklund transformations for P_4 and, hence, to see if it is possible to obtain all hierarchies of exact solutions of P_4.

Another direction for future research will be to evaluate the connection, if possible, between our 3×3 systems and other alternate Lax pair representations for P_4 and P_5. Thus, for instance, it may be possible to establish a link between the 3×3 systems found here and the order three and order four linear systems associated with the symmetric Painlevé equations found by Noumi *et al* [9]–[10]. Similarly, the connection between our monodromy Lax representation and the 2×2 Lax pair for P_4 found by Kitaev (see [22], [23]) and later by Milne *et al* [24] is yet to be established.

Acknowledgements This work was supported by ARC DP grant number DP0559019. The research was carried out during AVK's visits to the School of Mathematics and Statistics at the University of Sydney, Australia.

A On Spectral Interpretation of the Bäcklund Transformations for P_5

We recall the Bäcklund transformation for P_5 was found in [25] (see also [26]):

$$\hat{y} = 1 - \frac{2\sqrt{-2\delta}ty}{ty' - \sqrt{2\alpha}y^2 + (\sqrt{2\alpha} - \sqrt{-2\beta} + t\sqrt{-2\delta})y + \sqrt{-2\beta}}, \tag{A.1}$$

$$\sqrt{2\hat{\alpha}} = \frac{1}{2}\left(\frac{\gamma}{\sqrt{-2\delta}} + 1 - \sqrt{-2\beta} - \sqrt{2\alpha}\right), \tag{A.2}$$

$$\sqrt{-2\hat{\beta}} = \frac{1}{2}\left(\frac{\gamma}{\sqrt{-2\delta}} - 1 + \sqrt{-2\beta} + \sqrt{2\alpha}\right), \tag{A.3}$$

$$\frac{\hat{\gamma}}{\sqrt{-2\hat{\delta}}} = \sqrt{-2\beta} - \sqrt{2\alpha}, \qquad \sqrt{-2\hat{\delta}} = \sqrt{-2\delta} \neq 0, \tag{A.4}$$

where $y = y(t)$ and $\hat{y} = \hat{y}(t)$ solve P_5 for the parameters α, β, γ, and δ and $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$, and $\hat{\delta}$, respectively. The important feature of this transformation is that the branches of the square roots in equations (A.1)–(A.4) can be taken arbitrary but the same in all the formulae.

Our goal here is to discuss the spectral interpretation of this transformation. For this purpose we use the Jimbo-Miwa [1] isomonodromy representation of P_5. Consider the following linear matrix ODE:

$$\frac{d\Psi}{d\lambda} = \left(\frac{t}{2}\sigma_3 + \frac{A_0}{\lambda} + \frac{A_1}{\lambda - 1}\right)\Psi. \tag{A.5}$$

Here $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and the matrices A_p $(p = 0, 1)$ are independent of λ. Consider the following parametrization of the matrices A_p,

$$A_0 = \begin{pmatrix} z + \frac{\theta_0}{2} & -u(z+\theta_0) \\ z/u & -z - \frac{\theta_0}{2} \end{pmatrix}, \quad A_1 = \begin{pmatrix} -z - \frac{\theta_0+\theta_\infty}{2} & uy\left(z + \frac{\theta_0-\theta_1+\theta_\infty}{2}\right) \\ -\frac{1}{uy}\left(z + \frac{\theta_0+\theta_1+\theta_\infty}{2}\right) & z + \frac{\theta_0+\theta_\infty}{2} \end{pmatrix}.$$

Then, the isomonodromy deformations of equation (A.5) with respect to t are governed by the following system of nonlinear ODEs, which we will call the Isomonodromy Deformation System (IDS)

$$t\frac{dy}{dt} = ty - 2z(y-1)^2 - (y-1)\left(\frac{\theta_0 - \theta_1 + \theta_\infty}{2}y - \frac{3\theta_0 + \theta_1 + \theta_\infty}{2}\right), \quad (A.6)$$

$$t\frac{dz}{dt} = yz\left(z + \frac{\theta_0 - \theta_1 + \theta_\infty}{2}\right) - \frac{1}{y}(z+\theta_0)\left(z + \frac{\theta_0 + \theta_1 + \theta_\infty}{2}\right), \quad (A.7)$$

$$t\frac{d}{dt}\log u = -2z - \theta_0 + y\left(z + \frac{\theta_0 - \theta_1 + \theta_\infty}{2}\right) + \frac{1}{y}\left(z + \frac{\theta_0 + \theta_1 + \theta_\infty}{2}\right). \quad (A.8)$$

In this system θ_ν ($\nu = 0,\ 1,\ \infty$) are complex constants considered as parameters. Excluding the function z from equations (A.6)–(A.7) one finds that the function y satisfies the fifth Painlevé equation (1.2) for the set of the coefficients (1.13).

We note that by rescaling t we may set the coefficients $\hat{\delta} = \delta = -1/2$, and hence we may further put

$$\sqrt{-2\delta} = \sqrt{-2\hat{\delta}} = \varepsilon = \pm 1$$

in equations (A.1)–(A.4). To take into account the possibility of different choices of branches of the square roots in equations (A.1)–(A.4) we introduce the parameters ε_1, ε_2, $\hat{\varepsilon}_1$, $\hat{\varepsilon}_1$, each taking the value ± 1, in the following way

$$\sqrt{2\alpha} = \varepsilon_1 \frac{\theta_0 - \theta_1 + \theta_\infty}{2}, \qquad \sqrt{-2\beta} = \varepsilon_2 \frac{\theta_0 - \theta_1 - \theta_\infty}{2}, \qquad \gamma = 1 - \theta_0 - \theta_1, \quad (A.9)$$

$$\sqrt{2\hat{\alpha}} = \hat{\varepsilon}_1 \frac{\hat{\theta}_0 - \hat{\theta}_1 + \hat{\theta}_\infty}{2}, \qquad \sqrt{-2\hat{\beta}} = \hat{\varepsilon}_2 \frac{\hat{\theta}_0 - \hat{\theta}_1 - \hat{\theta}_\infty}{2}, \qquad \hat{\gamma} = 1 - \hat{\theta}_0 - \hat{\theta}_1. \quad (A.10)$$

By substituting equations (A.9) and (A.10) into formulae (A.2)–(A.4) we get the following equations relating the formal monodromies:

$$\hat{\theta}_\infty = \varepsilon\frac{\hat{\varepsilon}_1 - \hat{\varepsilon}_2}{2}(1 - \theta_0 - \theta_1) + \frac{\hat{\varepsilon}_1 + \hat{\varepsilon}_2}{2}\left(1 - \frac{\varepsilon_1 + \varepsilon_2}{2}(\theta_0 - \theta_1) - \frac{\varepsilon_1 - \varepsilon_2}{2}\theta_\infty\right), \quad (A.11)$$

$$\hat{\theta}_0 - \hat{\theta}_1 = \varepsilon\frac{\hat{\varepsilon}_1 + \hat{\varepsilon}_2}{2}(1 - \theta_0 - \theta_1) + \frac{\hat{\varepsilon}_1 - \hat{\varepsilon}_2}{2}\left(1 - \frac{\varepsilon_1 + \varepsilon_2}{2}(\theta_0 - \theta_1) - \frac{\varepsilon_1 - \varepsilon_2}{2}\theta_\infty\right), \quad (A.12)$$

$$\hat{\theta}_0 + \hat{\theta}_1 = 1 + \varepsilon\left(\frac{\varepsilon_1 - \varepsilon_2}{2}(\theta_0 - \theta_1) + \frac{\varepsilon_1 + \varepsilon_2}{2}\theta_\infty\right). \quad (A.13)$$

Equations (A.11)-(A.13) define $2^5 = 32$ different relations for the formal monodromies $\hat{\theta}$'s, according to the number of tuples $(\varepsilon, \varepsilon_1, \varepsilon_2, \hat{\varepsilon}_1, \hat{\varepsilon}_2)$. It is easy to notice that all these formulae can be presented as the compositions of the actions on the θ-parameters of the Schlesinger transformations "dressing" the infinity and zero points:

$$S_{\pm,\pm}: \quad \theta_\infty \to \theta_\infty \pm 1, \qquad \theta_0 \to \theta_0 \pm 1, \qquad \theta_1 \to \theta_1, \quad (A.14)$$

with possibly the reflections:

$$
\begin{aligned}
R_0: &\quad \theta_0 \to -\theta_0, \qquad \theta_1 \to \theta_1, \qquad \theta_\infty \to \theta_\infty, && \text{(A.15)}\\
R_1: &\quad \theta_0 \to \theta_0, \qquad \theta_1 \to -\theta_1, \qquad \theta_\infty \to \theta_\infty, && \text{(A.16)}\\
R_\infty: &\quad \theta_0 \to \theta_0, \qquad \theta_1 \to \theta_1, \qquad \theta_\infty \to -\theta_\infty, && \text{(A.17)}\\
R_{01}: &\quad \theta_0 \to \theta_1, \qquad \theta_1 \to \theta_0, \qquad \theta_\infty \to \theta_\infty, && \text{(A.18)}
\end{aligned}
$$

and the following Okamoto-like transformations, mixing the θ-variables:

$$
\mathcal{O} := \hat{\theta}_0 = \pm\frac{\theta_0 + \theta_1 - \theta_\infty}{2}, \quad \hat{\theta}_1 = \mp\frac{\theta_0 + \theta_1 + \theta_\infty}{2}, \quad \hat{\theta}_\infty = \theta_1 - \theta_0. \tag{A.19}
$$

B Similarity reduction to the third Painlevé equation

The following similarity reduction was derived independently in [6] and [7] and transforms system (1.10) into a system of ODEs that are solvable in terms of solutions of the generic P_3 equation

$$
\mathrm{P}_3: \quad \frac{d^2y}{dt^2} = \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{1}{t}\Big(\alpha y^2 + \beta\Big) + \gamma y^3 + \frac{\delta}{y}, \tag{B.1}
$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary complex parameters. The similarity reduction is given by

$$
\begin{aligned}
v_1 &= \exp[-\tfrac{1}{2}x_3 + i\rho_1 x_3]u_1\\
v_2 &= \exp[-\tfrac{1}{2}x_3 + i\rho_2 x_3]u_2\\
v_3 &= (x_1 - x_2)^{1-i\rho_3}u_3, \quad \tau = (x_1 - x_2)e^{x_3},
\end{aligned} \tag{B.2}
$$

where ρ_1, ρ_2, ρ_3 are real constants related by $\rho_1 + \rho_2 + \rho_3 = 0$. Under this reduction system (1.10) becomes

$$
\begin{aligned}
\tau^{1+i\rho_3}v_1' &= iv_2^* v_3^*\\
\tau^{1+i\rho_3}v_2' &= -iv_3^* v_1^*\\
\tau^{i\rho_3}v_3' &= iv_1^* v_2^*.
\end{aligned} \tag{B.3}
$$

It was shown in [7] that solutions of this system can be represented in terms of an SD-type equation that is equivalent to the particular case of the fifth Painlevé equation (1.2) with $\delta = 0$. It is well known [25] that equation (1.2) with δ taken to be zero is equivalent to the general P_3 equation. As in the case of P_4, P_5 and P_6 we will construct a monodromy Lax pair for the reduced system (B.3) and then carry out the explicit integration in terms of P_3.

We introduce the monodromy variable λ as follows

$$
\lambda = e^{-x_3}k, \tag{B.4}
$$

and then, taking $\kappa_3 = 0$ in (2.2) and writing $\Psi(x_j, k) = R(x_j)\Phi(\tau, \lambda)$ where $R(x_j)$ is given by

$$
R(x_1, x_2, x_3) = \operatorname{diag}\Big(\exp[-i\rho_2 x_3], \exp[i\rho_1 x_3], \exp[-\tfrac{1}{2}x_3]\Big),
$$

we obtain the following monodromy Lax pair

$$M\Phi_\lambda = \Big(Q^{(1)} + \frac{Q^{(0)}}{\lambda}\Big)\Phi \tag{B.5a}$$

$$M\Phi_\tau = \Big(\lambda P^{(1)} + P^{(0)}\Big)\Phi, \tag{B.5b}$$

where the matrices $M, Q^{(1)}, P^{(1)}, Q^{(0)}, P^{(0)}$ are given by

$$\begin{aligned} M &= \mathrm{diag}\,\big(1,1,0\big),\\ Q^{(1)} &= i\mathrm{diag}\,\big(-\tau\kappa_2, \tau\kappa_1, \kappa_1+\kappa_2\big),\\ P^{(1)} &= i\mathrm{diag}\,\big(-\kappa_2, \kappa_1, 0\big), \end{aligned}$$

and

$$\begin{aligned} Q^{(0)} &= i\begin{pmatrix} -\rho_2 & -\tau^{-i\rho_3}v_3^* & -v_2\\ \tau^{i\rho_3}v_3 & \rho_1 & v_1^*\\ -v_2^* & v_1 & 0\end{pmatrix},\\ P^{(0)} &= i\begin{pmatrix} 0 & \tau^{-1-i\rho_3}v_3^* & 0\\ \tau^{-1+i\rho_3}v_3 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}. \end{aligned}$$

We compare (B.5) with the following 3×3 system

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\Phi_\lambda = \left(\begin{pmatrix}t/2&0&0\\0&-t/2&0\\0&0&-1\end{pmatrix} + \frac{1}{\lambda}\begin{pmatrix}-\theta_\infty/2 & -\tilde{w}_3 & -w_2\\ w_3 & \theta_\infty/2 & -\tilde{w}_1\\ \tilde{w}_2 & w_1 & 0\end{pmatrix}\right)\Phi \tag{B.6a}$$

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\Phi_t = \left(\begin{pmatrix}\lambda/2&0&0\\0&-\lambda/2&0\\0&0&0\end{pmatrix} + \frac{1}{t}\begin{pmatrix}0 & -\tilde{w}_3 & 0\\ w_3 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right)\Phi, \tag{B.6b}$$

where $\{w_j, \tilde{w}_j\}$ are functions of t and θ_∞ is an arbitrary constant. The compatibility condition for (B.6) is

$$\begin{aligned} tw_1' &= \tilde{w}_2\tilde{w}_3, & t\tilde{w}_1' &= w_2w_3,\\ tw_2' &= -\tilde{w}_1\tilde{w}_3, & t\tilde{w}_2' &= -w_1w_3,\\ tw_3' &= -\theta_\infty w_3 + t\tilde{w}_1\tilde{w}_2, & t\tilde{w}_3' &= \theta_\infty\tilde{w}_3 + tw_1w_2. \end{aligned}$$

We note that the third row of (B.6a) gives the relation

$$\lambda\Phi_3 = \tilde{w}_2\Phi_1 + w_1\Phi_2,$$

and so we can eliminate Φ_3 from the above system. The resulting 2×2 system has the form:

$$\frac{d\phi}{d\lambda} = \left[\frac{t}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix} + \frac{1}{\lambda}\begin{pmatrix}-\theta_\infty/2 & -\tilde{w}_3\\ w_3 & \theta_\infty/2\end{pmatrix} - \frac{1}{\lambda^2}\begin{pmatrix}w_2\tilde{w}_2 & w_1w_2\\ \tilde{w}_1\tilde{w}_2 & w_1\tilde{w}_1\end{pmatrix}\right]\phi, \tag{B.7a}$$

$$\frac{d\phi}{dt} = \left[\frac{\lambda}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix} + \frac{1}{t}\begin{pmatrix}0 & -\tilde{w}_3\\ w_3 & 0\end{pmatrix}\right]\phi, \quad \phi = \begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}. \tag{B.7b}$$

Making the change of variables $t\mapsto t^2, \lambda\mapsto\lambda/t$ we find that system (B.7) is equivalent (up to gauge transformation) to the Jimbo–Miwa system for P_3 given in [1]

Written in terms of the new variables the compatibility condition becomes

$$\begin{aligned} tw_1' &= 2\tilde{w}_2\tilde{w}_3, & t\tilde{w}_1' &= 2w_2w_3, \\ tw_2' &= -2\tilde{w}_1\tilde{w}_3, & t\tilde{w}_2' &= -2w_1w_3, \\ tw_3' &= -2\theta_\infty w_3 + 2t^2\tilde{w}_1\tilde{w}_2, & t\tilde{w}_3' &= 2\theta_\infty\tilde{w}_3 + 2t^2 w_1w_2, \end{aligned} \tag{B.8}$$

for which we get first integrals

$$\begin{aligned} &w_1\tilde{w}_1 + w_2\tilde{w}_2 = c_1, \\ &w_1w_2w_3 - \tilde{w}_1\tilde{w}_2\tilde{w}_3 + \frac{1}{2}\theta_\infty\big(w_1\tilde{w}_1 - w_2\tilde{w}_2\big) = c_2, \end{aligned}$$

where c_1 and c_2 are constants. Without loss of generality we set $c_1 = 1$ and $c_2 = \theta_0/2$. Elementary computation now shows that the functions y and $\tilde{y}$, given by

$$y(t) = \frac{\tilde{w}_3}{tw_1w_2}, \quad \tilde{y}(t) = \frac{w_3}{t\tilde{w}_1\tilde{w}_2}, \tag{B.9}$$

satisfy the classical P_3 equation with $\alpha = 4\theta_0, \beta = 4(1-\theta_\infty), \gamma = 4, \delta = -4$ for the y equation, and $\tilde{\alpha} = 4\theta_0, \tilde{\beta} = 4(1+\theta_\infty), \tilde{\gamma} = 4, \tilde{\delta} = -4$ for the $\tilde{y}$ equation.

In order to solve the reduced 3WRI system (B.3) in terms of P_3 we compare the monodromy system (B.5) with system (B.7) to find

$$v_1(\tau) = w_1(\tau), \quad v_2(\tau) = w_2(\tau), \quad v_3(\tau) = \tau^{-i\rho_3}w_3(\tau), \tag{B.10a}$$

$$v_1^*(\tau) = \tilde{w}_1(\tau), \quad v_2^*(\tau) = \tilde{w}_2(\tau), \quad v_3^*(\tau) = \tau^{i\rho_3}\tilde{w}_3(\tau). \tag{B.10b}$$

We will show that the functions $\{w_j(\tau), \tilde{w}_j(\tau)\}$ can be expressed in terms of $y(t)$, $\tilde{y}(t)$, $z(t)$ and $w(t)$, where $z = tw_1\tilde{w}_1$, $w = w_1w_2$ and $\tau = t^2$. Using the expression for y given in (B.9) and the compatibility conditions (B.8), we obtain the following system for $\{y, z, w\}$

$$t\frac{dy}{dt} = (4z - 2t)y^2 + (2\theta_\infty - 1)y + 2t, \tag{B.11a}$$

$$t\frac{dz}{dt} = 4z(t-z)y - (2\theta_\infty - 1)z + (\theta_0 + \theta_\infty)t, \tag{B.11b}$$

$$t\frac{d}{dt}(\ln w) = -(4z - 2t)y. \tag{B.11c}$$

It follows from (B.8) and the above expressions that the functions $\{w_j, \tilde{w}_j\}$ are given as

$$w_1(t) = f\left(\frac{z}{t-z}\right)^{1/2}, \quad \tilde{w}_1(t) = \frac{1}{ft}\Big(z(t-z)\Big)^{1/2} \tag{B.12a}$$

$$w_2(t) = \frac{1}{g}\left(\frac{t-z}{z}\right)^{1/2}, \quad \tilde{w}_2(t) = \frac{g}{t}\Big(z(t-z)\Big)^{1/2}, \tag{B.12b}$$

where f and g are arbitrary functions such that $w = f/g$, and

$$t\frac{d}{dt}(\ln w_3) = \frac{2t}{\tilde{y}} - 2\theta_\infty, \tag{B.12c}$$

$$t\frac{d}{dt}(\ln \tilde{w}_3) = \frac{2t}{y} + 2\theta_\infty. \tag{B.12d}$$

To conclude this appendix we state without proof an alternate reduction of system (B.6) to a 2×2 system. Using the generalized Laplace transform (3.11) in (B.6a), the resulting matrix equation has the form

$$\begin{pmatrix} x - t/2 & 0 & 0 \\ 0 & x + t/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{d\tilde{Y}}{dx} = - \begin{pmatrix} -\theta_\infty/2 + 1 & -\tilde{w}_3 & -w_2 \\ w_3 & \theta_\infty/2 + 1 & -\tilde{w}_1 \\ \tilde{w}_2 & w_1 & 0 \end{pmatrix} \tilde{Y}. \tag{B.13}$$

By choosing the determinant of the RHS matrix to be zero we can make a gauge transformation $\tilde{Y} = G\hat{Y}$ where G is the diagonalizing matrix, to obtain

$$\frac{d\hat{Y}}{dx} = \left[\hat{A}_2 + \frac{1}{x - t/2}\hat{A}_1 + \frac{1}{x + t/2}\hat{A}_0 \right] \hat{Y}, \tag{B.14}$$

where the $\hat{A}_j$ are all of the form

$$\hat{A}_j = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 0 \end{pmatrix}.$$

This can then be reduced (after a change of variables) to a 2×2 system of the form

$$\frac{dY}{dx} = \left[A_2 + \frac{1}{x - t}A_1 + \frac{1}{x}A_0 \right] Y. \tag{B.15}$$

Isomonodromic deformations in t of this equation are parametrized by solutions of the degenerate fifth Painlevé equation (1.2) with $\delta = 0$, see [27], [28].

The Lax pair (B.7) has two irregular singularities at zero and infinity, while the Lax pair (B.14) has two regular singularities at zero and t and an irregular singularity at infinity. By comparing the two reductions given above it is possible to obtain a mapping between these two systems and, as with the Bäcklund transformation for P5, a mapping between two solutions of P_3.

References

[1] M. Jimbo, T. Miwa, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II*, Phys. D, **2** (1981) 407–448.

[2] M. Jimbo, T. Miwa, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. III*, Phys. D, **4** (1981) 26–46.

[3] M. Jimbo, T. Miwa, K. Ueno, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I*, Phys. D, **2** (1981) 306–352.

[4] J. Harnad, *Dual isomonodromic deformations and moment maps to loop algebras*, Comm. Math. Phys., **166** (1994) 337–365.

[5] M. Mazzocco, *Painlevé sixth equation as isomonodromic deformations equation of an irregular system*, in "The Kowalevski property", CRM Proc. Lecture Notes vol 32, (2002, Providence, RI) 219–238.

[6] A.V. Kitaev, *On similarity reductions of the three-wave resonant system to the Painlevé equations*, J. Phys. A: Math. Gen., **23** (1990) 3543–3553.

[7] L. Martina, P. Winternitz, *Analysis and applications of the symmetry group of the multidimensional three-wave resonant interaction problem*, Ann. Phys., **196** (1989) 231–277.

[8] F.J. Bureau, A. Garcet, J. Goffar, *Transformées algébriques des équations du second ordre dont l'intégrale générale est à points critiques fixes*, Ann. Mat. Pura Appl., **92** (1972) 177–191.

[9] M. Noumi, Y. Yamada, *Symmetries in the fourth Painlevé equation and Okamoto polynomials*, Nagoya Math. J., **153** (1999) 53–86.

[10] M. Noumi, Y. Yamada, *Affine Weyl group symmetries in Painlevé type equations*, in "Towards the exact WKB analysis of differential equations, linear or nonlinear". Kyoto Univ. Pr. (2000) 245–259.

[11] A. Sen, A.N.W. Hone, P.A. Clarkson, *Darboux transformations and the symmetric fourth Painlevé equation*, J. Phys. A: Math. Gen., **38** (2005) 9751–9764.

[12] A. Sen, A.N.W. Hone, P.A. Clarkson, *On Lax pairs of the symmetric Painlevé equations*, (preprint).

[13] R. Willox, J. Hietarinta, *Painlevé equations from Darboux chains: I.* $P_{III} - P_V$, J. Phys. A: Math. Gen., **36** (2003) 10615–10635.

[14] R. Conte, A.M. Grundland, M. Musette, *A reduction of the resonant three-wave interaction to the generic sixth Painlevé equation*, J. Phys. A: Math. Gen., **39** (2006) 12115–12127.

[15] S. Kakei, T. Kikuchi, *The sixth Painlevé equation as similarity reduction of GL3 hierarchy*, nlin.SI/0508021 (preprint).

[16] P.P. Boalch, *From Klein to Painlevé via Fourier, Laplace and Jimbo*, Proc. London. Math. Soc., **90** (2005) 167–208.

[17] P.P. Boalch, *Six results on Painlevé VI*, math.AG/0503043 (preprint).

[18] D.J. Benney, A.C. Newell, *The propagation of nonlinear wave envelopes*, J. Math. and Phys., **46** (1967) 133–139.

[19] V.E. Zakharov, S.V. Manakov, *The theory of resonance interaction of wave packets in nonlinear media*, Sov. Phys. JETP, **42** (1975) 842–850.

[20] D.J. Kaup, *The inverse scattering solution for the full three-dimensional three-wave resonant interaction*, Phys. D, **1** (1980) 45–67.

[21] A.S. Fokas and M.J. Ablowitz, *On the inverse scattering transform of multidimensional nonlinear equations related to first-order systems in the plane*, J. Math. Phys., **25** (1984) 2494–2505.

[22] A.V. Kitaev, *Self-similar solutions of the modified nonlinear Schrödinger equation*, Theor. Math. Phys., **64** (1985) 878–894.

[23] A.R. Its, V. Yu-Novokshenov, *The Isomonodromic Deformation Method in the Theory of Painlevé Equations*, Lect. Notes Phys. 1191 (Berlin, Springer, 1986).

[24] A.E. Milne, P.A. Clarkson, A.P. Bassom, *Application of the isomonodromy deformation method to the fourth Painlevé equation*, Inv. Problems, **13** (1997) 421–439.

[25] V.I. Gromak, *On the theory of Painlevé's equations*, Diff. Uravnen., **11** (1975) 373–376.

[26] V.I. Gromak, N.A. Lukashevich, *Special classes of solutions of Painlevé equations*, Diff. Uravnen., **18** (1982) 419–429.

[27] A.V. Kitaev, *Quadratic transformations for the third and fifth Painlevé equations*, J. Math. Sci., **136** (2006) 3586–3595.

[28] A.A. Kapaev, E. Hubert, *A note on the Lax pairs for Painlevé equations*, J. Phys. A: Math. Gen., **32** (1999) 8145–8156.