PERIODIC ELEMENTS OF THE FREE IDEMPOTENT GENERATED SEMIGROUP ON A BIORDERED SET

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Dedicated to the memory of Bret Tilson

Abstract. We show that every periodic element of the free idempotent generated semigroup on an arbitrary biordered set belongs to a subgroup of the semigroup.

The *biordered set of a semigroup* S is the set of idempotents of S considered as a partial groupoid with respect to the restriction of the multiplication of S to those pairs (e, f) of idempotents such that $ef = e$, $ef = f$, $fe = e$ or $fe = f$. Nambooripad [6] who has initiated an axiomatic approach to biordered sets has defined an *abstract biordered set* as a partial groupoid satisfying certain second order axioms. The first author [3] has confirmed the adequacy of Nambooripad's axiomatization by showing that each abstract biordered set is in fact the biordered set of a suitable semigroup. Namely, if $\langle E, \circ\rangle$ is an abstract biordered set, denote by $IG(E)$ the semigroup with presentation

$$IG(E) = \{E \mid ef = e \circ f \text{ whenever } e \circ f \text{ is defined in } E\}.$$

The semigroup $IG(E)$ is called the *free idempotent generated semigroup on* E. In [3] it has been shown that the biordered set of $IG(E)$ coincides with the initial biordered set $\langle E, \circ\rangle$ (see Lemma 2 below for a precise formulation of this result).

The structure of the free idempotent generated semigroup on a biordered set is not yet well understood. It was conjectured that subgroups of such a semigroup should be free. Though confirmed for some partial cases (see [5, 7, 8, 9]), this conjecture has been recently disproved by Brittenham, Margolis, and Meakin [1] who have found a biordered set $\langle E, \circ\rangle$ such that the semigroup $IG(E)$ has the free abelian group of rank 2 among its subgroups. Moreover, in the subsequent paper [2] the same authors have proved that if F is any field, and $E_3(F)$ is the biordered set of the monoid of all 3×3 matrices over F, then the free idempotent generated semigroup over $E_3(F)$ has a subgroup isomorphic to the multiplicative group of F. In particular, letting

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F be the field of complex numbers, one concludes that the free idempotent generated semigroup on a biordered set can contain group elements of any finite order.

Recall that an element a of a semigroup S is said to be *periodic* if a generates a finite subsemigroup in S; in other words, if

$$a^h = a^{h+d} \tag{1}$$

for some positive integers h and d. Given a, the least h and d verifying the equality (1) are called respectively the *index* and the *period* of a. The aforementioned discovery by Brittenham, Margolis, and Meakin [2] shows that there is no restriction to periods of periodic elements in the free idempotent generated semigroup on a biordered set. The main result of the present note demonstrates that, in contrast, indices of periodic elements in such a semigroup are severely restricted, namely, they must be equal to 1. In other words, we aim to show that every periodic element of $IG(E)$ must belong to a subgroup of $IG(E)$.

We assume the reader's familiarity with Green's relations $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$ and their basic properties that can be found in the early chapters of any general semigroup theory text, e.g., [4, Chapter 2]. The following property is also elementary but perhaps less known.

Lemma 1. *Let S be a semigroup, $a, e \in S$, $e^2 = e$, p, q positive integers where $p \leq q$. Then $a^p \mathrel{\mathscr{R}} a^q = e$ implies $a^p \mathrel{\mathscr{H}} e$.*

Proof. Clearly, $e = a^{q-p}a^p \in S^1a^p$. Since $a^p = eb$ for some $b \in S^1$, we have

$$a^pe = a^{p+q} = ea^p = e(eb) = eb = a^p.$$

Thus, $a^p \in S^1e$, whence $a^p \mathrel{\mathscr{L}} e$ and $a^p \mathrel{\mathscr{H}} e$. □

We fix an arbitrary biordered set $\langle E, \circ\rangle$. Now let E^+ be the free semigroup on E and $\varphi : E^+ \to IG(E)$ the onto morphism extending the identity map on E.

Lemma 2. *If $w \in E^+$ and $w\varphi$ is idempotent, then $w\varphi = e\varphi$ for some $e \in E$.*

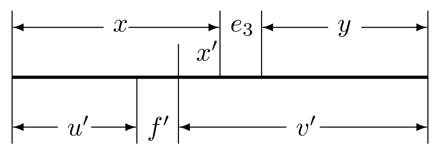
Proof. This is the main result of [3]. □

As usual, E^* stands for E^+ with the empty word 1 adjoined.

Lemma 3. *If $w \in E^+$ and $w\varphi = e\varphi$ for some $e \in E$, then there exist $u, v \in E^*$ and $f \in E$ such that $w = ufv$ and $(uf)\varphi \mathrel{\mathscr{L}} f\varphi \mathrel{\mathscr{R}} (fv)\varphi$.*

Proof. Let $\sigma = \ker\varphi$. Clearly, every two σ-related words in E^+ can be connected by a sequence of elementary σ-transitions of the form $xe_1e_2y \to xe_3y$ or $xe_3y \to xe_1e_2y$ where $x, y \in E^*$, $e_1, e_2, e_3 \in E$ and $e_1 \circ e_2 = e_3$ in the biordered set $\langle E, \circ\rangle$. We induct on the minimum length n of such a sequence from w to e. If $n = 0$, that is $w = e$, the claim is obvious since we can set $u = v = 1$ and $f = e$. Suppose $n > 0$ and let $w \to w'$ be the first σ-transition in a sequence of minimum length connecting w and e. By the induction assumption, $w' = u'f'v'$ for some $u', v' \in E^*$ and $f' \in E$ such that $(u'f')\varphi \mathrel{\mathscr{L}} f'\varphi \mathrel{\mathscr{R}} (f'v')\varphi$. On the other hand, for some $x, y \in E^*$, $e_1, e_2, e_3 \in E$, we have the decompositions $w = xe_1e_2y$, $w' = xe_3y$ (the contraction case) or $w = xe_3y$, $w' = xe_1e_2y$ (the expansion case).

Consider the contraction case. We have $w' = xe_3y = u'f'v'$. First suppose that the distinguished occurrence of f' happens within the word x, that is $x = u'f'x'$, $v' = x'e_3y$ for some $x' \in E^*$:



Then the word $w = xe_1e_2y$ also decomposes as $w = ufv$ where $u = u'$, $f = f'$, and $v = x'e_1e_2y$ so that

$$(uf)\varphi = (u'f')\varphi \mathrel{\mathscr{L}} f'\varphi = f\varphi$$

and

$$f\varphi = f'\varphi \mathrel{\mathscr{R}} (f'v')\varphi = (f'x'e_3y)\varphi = (fx'e_1e_2y)\varphi = (fv)\varphi.$$

Thus,

$$(uf)\varphi \mathrel{\mathscr{L}} f\varphi \mathrel{\mathscr{R}} (fv)\varphi,$$

as required.

The situation when the distinguished occurrence of f' happens within the word y is handled in a symmetric way.

Now suppose that $x = u'$, $y = v'$ and $e_3 = f'$. Then $f' = e_1 \circ e_2$ in the biordered set $\langle E, \circ \rangle$. By the definition of a biordered set, the product $e_1 \circ e_2$ is defined if and only if either 1) $e_1 \circ e_2 = e_1$, or 2) $e_1 \circ e_2 = e_2$, or 3) $e_2 \circ e_1 = e_1$, or 4) $e_2 \circ e_1 = e_2$. In Cases 1 and 3 set $u = u' = x$ and $v = e_2y = e_2v'$. Then $w = ue_1v$. Since $(u'f')\varphi \mathrel{\mathscr{L}} f'\varphi \mathrel{\mathscr{R}} (f'v')\varphi$ and $f'\varphi = (e_1e_2)\varphi$, we have

$$(ue_1e_2)\varphi \mathrel{\mathscr{L}} (e_1e_2)\varphi \mathrel{\mathscr{R}} (e_1v)\varphi.$$

Under the condition of each of the cases under consideration, $(e_1e_2e_1)\varphi = e_1\varphi$ whence $e_1\varphi \mathrel{\mathscr{R}} (e_1e_2)\varphi$. Multiplying the relation $(ue_1e_2)\varphi \mathrel{\mathscr{L}} (e_1e_2)\varphi$ through on the right by $e_1\varphi$, we get $(ue_1)\varphi \mathrel{\mathscr{L}} e_1\varphi$. Thus,

$$(ue_1)\varphi \mathrel{\mathscr{L}} e_1\varphi \mathrel{\mathscr{R}} (e_1v)\varphi,$$

as required. Cases 2 and 4 are dual.

Now consider the expansion case. We have $w' = xe_1e_2y = u'f'v'$. The situations when the distinguished occurrence of f' happens within x or y are completely similar to the analogous situations in the contraction case. Suppose that $x = u'$, $e_1 = f'$ and $e_2y = v'$. Then we set $u = u' = x$ and $v = y$, whence $w = ue_3v$. Since $(u'f')\varphi \mathrel{\mathscr{L}} f'\varphi \mathrel{\mathscr{R}} (f'v')\varphi$, we have

$$(ue_1)\varphi \mathrel{\mathscr{L}} e_1\varphi \mathrel{\mathscr{R}} (e_1e_2v)\varphi = (e_3v)\varphi.$$

Multiplying the relation $(ue_1)\varphi \mathrel{\mathscr{L}} e_1\varphi$ through on the right by $e_2\varphi$, we obtain $(ue_3)\varphi = (ue_1e_2)\varphi \mathrel{\mathscr{L}} (e_1e_2)\varphi = e_3\varphi$. On the other hand, from the relation

$$e_1\varphi \mathrel{\mathscr{R}} (e_3v)\varphi \tag{2}$$

we have $e_1\varphi = (e_3v)\varphi \cdot s = e_3\varphi \cdot (v\varphi \cdot s)$ for some $s \in IG(E)$, and since $e_3\varphi = (e_1e_2)\varphi = e_1\varphi \cdot e_2\varphi$, we conclude that $e_3\varphi \mathrel{\mathscr{R}} e_1\varphi$. From this and from (2) we get $e_3\varphi \mathrel{\mathscr{R}} (e_3v)\varphi$. Thus,

$$(ue_3)\varphi \mathrel{\mathscr{L}} e_3\varphi \mathrel{\mathscr{R}} (e_3v)\varphi,$$

as required. The situation when $x = u'e_1$, $e_2 = f'$ and $y = v'$ is handled in a symmetric way. □

We are ready to state and to prove our main result.

Theorem. *Let $\langle E, \circ\rangle$ be a biordered set, $IG(E)$ the free idempotent generated semigroup on E. Every periodic element of $IG(E)$ lies in a subgroup of $IG(E)$.*

Proof. Let $w = e_1 \cdots e_n$, where $e_1, \dots, e_n \in E$, be a word in E^+ such that $w\varphi \in IG(E)$ is periodic. Then $(w\varphi)^k = w^k\varphi$ is idempotent for some k, whence, by Lemma 2, $w^k\varphi = e\varphi$ for some $e \in E$. If $k = 1$, there is nothing to prove, so we suppose $k > 1$ and apply Lemma 3 to w^k. It yields a decomposition of the form

$$w^k = (e_1 \cdots e_n)^\ell e_1 \cdots e_{i-1} \cdot e_i \cdot e_{i+1} \cdots e_n (e_1 \cdots e_n)^m$$

such that $0 \le \ell, m < k$, $1 \le i \le n$, and

$$\big((e_1 \cdots e_n)^\ell e_1 \cdots e_{i-1}e_i\big)\varphi \mathrel{\mathscr{L}} e_i\varphi \mathrel{\mathscr{R}} \big(e_ie_{i+1} \cdots e_n(e_1 \cdots e_n)^m\big)\varphi. \tag{3}$$

Using Green's lemma, we deduce from (3) the following relations:

$$\begin{tikzcd}
w^{\ell+1}\varphi \arrow[r, dash, "\mathscr{R}"] \arrow[d, dash, "\mathscr{L}"] & (w^\ell e_1 \cdots e_{i-1}e_i)\varphi \arrow[r, dash, "\mathscr{R}"] \arrow[d, dash, "\mathscr{L}"] & w^k\varphi = e\varphi \arrow[d, dash, "\mathscr{L}"] \\
(e_ie_{i+1} \cdots e_n)\varphi \arrow[r, dash, "\mathscr{R}"] \arrow[d, dash, "\mathscr{L}"] & e_i\varphi \arrow[r, dash, "\mathscr{R}"] \arrow[d, dash, "\mathscr{L}"] & (e_ie_{i+1} \cdots e_nw^m)\varphi \arrow[d, dash, "\mathscr{L}"] \\
w\varphi \arrow[r, dash, "\mathscr{R}"] & (e_1 \cdots e_{i-1}e_i)\varphi \arrow[r, dash, "\mathscr{R}"] & w^{m+1}\varphi
\end{tikzcd}$$

(The "initial" relations in (3) are represented by the bold lines.) In particular, $w^{\ell+1}\varphi \mathrel{\mathscr{R}} w^k\varphi$. Since $\ell+1 \le k$, we can apply Lemma 1 with $a = w\varphi$, $p = \ell+1$ and $q = k$, thus obtaining $w^{\ell+1}\varphi \mathrel{\mathscr{H}} e\varphi$. Hence $w\varphi \mathrel{\mathscr{L}} e\varphi$ and the dual of Lemma 1 implies that $w\varphi \mathrel{\mathscr{H}} e\varphi$, that is, $w\varphi$ belongs to a subgroup of $IG(E)$. □

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generated semigroup on a biordered set may have non-idempotent periodic elements, and publishing a result about objects that might not exist did not seem to be justified. It was not until very recently that the examples in [2] have confirmed that our theorem has indeed a non-void applicability range.

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