FINITENESS CONDITIONS IN COVERS OF POINCARÉ DUALITY SPACES

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Abstract. A closed 4-manifold (or, more generally, a finite PD_4-space) has a finitely dominated infinite regular covering space if and only if either its universal covering space is finitely dominated or it is finitely covered by the mapping torus of a self homotopy equivalence of a PD_3-complex.

A Poincaré duality space is a space X of the homotopy type of a cell complex which satisfies Poincaré duality with local coefficients. It is finite if the singular chain complex of the universal cover $\widetilde{X}$ is chain homotopy equivalent to a finite free $\mathbb{Z}[\pi_1(X)]$-complex. (The PD-space X is homotopy equivalent to a Poincaré duality complex $\Leftrightarrow$ it is finitely dominated $\Leftrightarrow$ $\pi_1(X)$ is finitely presentable. See [2].)

In this note we show that finiteness hypotheses in two theorems about covering spaces of PD-complexes may be relaxed. Theorem 4 extends a criterion of Stark to all Poincaré duality groups. The main result is Theorem 5, which characterizes finite PD_4-spaces with finitely dominated infinite regular covering spaces.

We shall often write "vPD-group" instead of "virtual Poincaré duality group", and similarly vPD_r, vFP, etc. We say also that a group G is a *weak PD_r-group* if $H^r(G;\mathbb{Z}[G]) \cong Z$ and $H^q(G;\mathbb{Z}[G]) = 0$ for $q \neq r$.

1. some lemmas

The following lemma is essentially from [5]. We shall use it in conjunction with universal coefficient spectral sequences.

Lemma 1. *Let G be a group and k be $\mathbb{Z}$ or a field, and let A be a $k[G]$-module which is free of finite rank m as a k-module. Then $Ext^q_{k[G]}(A, k[G]) \cong (H^q(G; k[G]))^m$ for all q.*

Proof. Let $(g\phi)(a) = g.\phi(g^{-1}a)$ for all $g \in G$ and $\phi \in Hom_k(A, k[G])$. Let $\{\alpha_i\}_{1\leq i\leq m}$ be a basis for A as a free k-module, and define a map

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$f : Hom_k(A, k[G]) \to k[G]^m$ by $f(\phi) = (\phi(\alpha_1), \ldots, \phi(\alpha_m))$ for all $\phi \in Hom_k(A, k[G])$. Then f is an isomorphism of left $k[G]$-modules. The lemma now follows, since $Ext^q_{k[G]}(A, k[G]) \cong H^q(G; Hom_k(A, k[G]))$. (See Proposition III.2.2 of [3].) $\square$

Lemma 2. *If $H^q(G;\mathbb{Z}[G])$ is 0 (respectively, finitely generated as an abelian group) for all $q \leq q_0$ and B is a $\mathbb{Z}[G]$-module which is finitely generated as an abelian group then $Ext^q_{\mathbb{Z}[G]}(B, \mathbb{Z}[G])$ is 0 (respectively, finitely generated as an abelian group) for all $q \leq q_0$.*

Proof. Let T be the $\mathbb{Z}$-torsion submodule of B, and let H be the kernel of the action of G on T. Then T is a finite $\mathbb{Z}[G/H]$-module, and so is a quotient of a finitely generated free $\mathbb{Z}[G/H]$-module A. Let A_1 be the kernel of the projection from A to T. Clearly A and A_1 are $\mathbb{Z}[G]$-modules which are free of (the same) finite rank as abelian groups. We now apply the long exact sequence of $Ext^*_{\mathbb{Z}[G]}(-, \mathbb{Z}[G])$ together with Lemma 1 to the short exact sequences $0 \to A_1 \to A \to T \to 0$ and $0 \to T \to B \to B/T \to 0$. $\square$

2. virtual Poincaré duality groups

Stark has shown that a finitely presentable group G of finite virtual cohomological dimension is a virtual Poincaré duality group if and only if it is the fundamental group of a closed PL manifold M whose universal cover $\widetilde{M}$ is homotopy finite [10]. The main step in showing the sufficiency of the latter condition involves showing first that G is of type vFP, and is established in [11]. If G_1 is an FP subgroup of finite index in G then $B = K(G_1, 1)$ is finitely dominated. Hence on applying the Gottlieb-Quinn Theorem to the fibration $\widetilde{M} \to M_1 \to B$ of the associated covering space M_1 it follows that $\widetilde{M}$ and B are Poincaré duality complexes. In particular, G_1 is a Poincaré duality group.

There are however Poincaré duality groups in every dimension $n \geq 4$ which are not finitely presentable. We shall give an analogue of Stark's sufficency result for such groups, using an algebraic criterion instead of the Gottlieb-Quinn Theorem. In the next two results we shall assume that M is a PD_n-space with fundamental group π, $F = M_\nu$ is the covering space associated to a normal subgroup ν of π, $G = \pi/\nu$ and k is $\mathbb{Z}$ or a field.

Lemma 3. *Suppose that $H_p(F;k)$ is finitely generated for all $p \leq [n/2]$. Then $H_p(F;k)$ is finitely generated for all p if and only if $H^q(G;k[G])$ is finitely generated as a k-module for $q \leq [(n-1)/2]$, and then $H^q(G;k[G])$ is finitely generated as k-module for all q. If*

$H^s(G;k[G])=0$ for $s<q$ then $H_{n-s}(F;k)=0$ for $s<q$ and $H_{n-q}(F;k)\cong H^q(G;k[G])$.

Proof. Let $E_2^{pq}=Ext^q_{k[G]}(H_p(M;k[G]),k[G])\Rightarrow H^{p+q}(M;k[G])$ be the Universal Coefficient spectral sequence for the equivariant cohomology of M. Then $E_2^{pq}=Ext^q_{k[G]}(H_p(F;k),k[G])$, while $H^{p+q}(M;k[G])\cong H_{n-p-q}(F;k)$, by Poincaré duality for M.

If $H^q(G;k[G])$ is finitely generated for $q\leq[(n-1)/2]$ then E_2^{pq} is finitely generated for all $p+q\leq[(n-1)/2]$, by Lemmas 1 and 2. Hence $H_p(F;k)$ is finitely generated for all $p\geq n-[(n-1)/2]$, and hence for all p. Conversely, if this holds and $H^s(G;k[G])$ is finitely generated for $s<q$ then E_r^{ps} is finitely generated for all $p\geq 0$, $r\geq 2$ and $s<q$. Since $H^q(M;k[G])\cong H_{n-q}(F;k)$ is finitely generated as a k-module it follows that $H^q(G;k[G])$ is finitely generated as a k-module. Hence $H^q(G;k[G])$ is finitely generated for all q.

The final assertion is an immediate consequence of duality and the UCSS. □

Theorem 4. *If $H_p(F;k)$ is finitely generated for all p then G is FP_∞ over k and $H^s(G;k[G])\neq 0$ for some $s\leq n$. If moreover $k=\mathbb{Z}$ and $v.c.d.G<\infty$ then G is virtually a PD_r-group, for some $r\leq n$.*

Proof. Let $C_*(\widetilde{M})$ be the equivariant chain complex of the universal covering space $\widetilde{M}$. Since M is a PD_n-space $C_*(\widetilde{M})$ is chain homotopy equivalent to a finite projective $\mathbb{Z}[\pi]$-complex. Hence $C_*(F;k)=k[G]\otimes_{\mathbb{Z}[\pi]}C_*(\widetilde{M})$ is chain homotopy equivalent to a finite projective $k[G]$-complex. The arguments of [11] apply equally well with coefficients k a field (instead of $\mathbb{Z}$), and thus the hypotheses of Lemma 3 imply that G is FP_∞ over k.

If $v.c.d.G<\infty$ we may assume without loss of generality that $c.d.G<\infty$, and so G is FP. Since $H_q(F;\mathbb{Z}))$ is finitely generated for all q the groups $H^s(G;\mathbb{Z}[G])$ are all finitely generated, and since $H_0(F;\mathbb{Z})=\mathbb{Z}$ we must have $H^s(G;\mathbb{Z}[G])\neq 0$ for some $s\leq n$, by Lemma 3. Then G is a PD_r-group, where $r=\min\{s\mid H^s(G;\mathbb{Z}[G])\neq 0\}\leq n$, by Theorem 3 of [5]. □

This complements the main result of [9], in which it is shown that if the $\mathbb{Z}[\nu]$-chain complex $C_*(\widetilde{F})=C_*(\widetilde{M})|_\nu$ has finite $[n/2]$-skeleton and G is a weak PD_r-group then F is a PD_{n-r}-space. If we drop the hypothesis "$v.c.d.G<\infty$" must G still be a weak PD_r-group for $r=\min\{s\mid H^s(G;\mathbb{Z}[G])\neq 0\}$? (In other words, if G is FP_∞, $H^s(G;\mathbb{Z}[G])=0$ for $s<r$ and $H^r(G;\mathbb{Z}[G])\cong\mathbb{Z}$ must G be a weak PD_r-group?) If $r\leq 2$ such a group is in fact a vPD_r-group (by the

main result of [1], when $r = 2$), but for each $n \geq 2$ and $k \geq \binom{n+1}{2}$ there are weak PD_k-groups which act freely and cocompactly on $S^{2n-1} \times \mathbb{R}^k$, but which are not virtually torsion-free [6]. Thus if $r \geq 6$ weak PD_r-groups need not be vPD_r-groups, and so the other conditions do not imply that $v.c.d.G < \infty$, in general. Little is known about the intermediate cases $r = 3, 4$ or 5. In paricular, it is an open question whether a group G of type FP_∞ such that $H^3(G; \mathbb{Z}[G]) \cong \mathbb{Z}$ is virtually a PD_3-group. (The fact that local homology manifolds which are homology 2-spheres are standard may be some slight evidence for this being true.)

Stark's argument for realization in the finitely presentable case can be adapted to show that any vPD_n-group acts freely on a 1-connected homotopy finite complex, with quotient a PD_m-space for some $m \geq n$. However finite presentability is needed in order to obtain a free *cocompact* action on a 1-connected complex. A natural converse to Theorem 4 (analogous to Stark's realization result) might be to show that every vPD group acts freely and cocompactly on some connected manifold X with $H_q(X; \mathbb{Z})$ finitely generated for all q. It would be enough to show that G acts freely on an m-complex Y such that $H_*(Y; \mathbb{Z})$ is finitely generated and $X = Y/G$ is a finite complex. For we may realize the homotopy type of X by a $2m$-dimensional handlebody M_o, and the closed manifold $DM = M_o \cup M_o$ has the desired properties, by Theorem 3 and Poincaré duality in DM. (Since every such G is FP_2, and hence is the quotient of a finitely presentable group by a perfect normal subgroup, by Exercise VIII.§5.3(b) of [3], we might also require that $H_1(Y; \mathbb{Z}) = 0$. Note also that if Y is 1-connected, finite-dimensional and $H_q(Y; \mathbb{Z})$ is finitely generated for all q then Y is homotopy finite.)

3. FINITELY DOMINATED COVERING SPACES

In [7] we showed that if a PD_4-complex M has a finitely dominated infinite regular covering space M_ν and $\nu = \pi_1(M_\nu)$ is FP_3 then either the universal covering space $\widetilde{M}$ is contractible or homotopy equivalent to S^2 or S^3, or M is the mapping torus of a self homotopy equivalence of a PD_3-complex. The hypothesis that ν be FP_3 was used at only one point (and is a consequence of M_ν being finitely dominated, if M is aspherical). We shall show here that we may assume instead that M be a finite PD_4-space, which is perhaps a more natural hypothesis, as it is satisfied by all closed 4-manifolds.

Theorem 5. *Let M be a finite PD_4-space with fundamental group π, and let ν be an infinite normal subgroup of π such that $G = \pi/\nu$ has one end and the associated covering space M_ν is finitely dominated. Then G is of type FP_∞ and M is aspherical.*

Proof. Let $F = M_\nu$ and let k be $\mathbb{Z}$ or a field. Then G is of type FP_∞ and $H^q(G;k[G])$ is finitely generated as a k-module for all q, by Lemma 3 and Theorem 4. Moreover $Ext^q_{k[\pi]}(H_p(F;k),k[\pi]) = 0$ for $q \leq 1$ and all p, since G has one end, and so $H_q(F;k) = 0$ for $q \geq 3$. In particular, $H^2(G;\mathbb{Z}[G]) \cong H_2(F;\mathbb{Z})$ is torsion-free, and so is a free abelian group of finite rank.

We may assume that F is not acyclic and G is not virtually a PD_2-group, by Theorem 3.9 of [7]. (Note that in the original version of [7] Theorem 3.9 was formulated in terms of PD_4-complexes. The arguments given there apply equally well to PD_4-spaces.) Therefore $H^2(G;k[G]) = 0$ for all k, by the main result of [1]. Hence $H_2(F;\mathbb{F}_p) = 0$ for all primes p, so $H_1(F;\mathbb{Z})$ is torsion-free and nonzero. Therefore $H^s(G;\mathbb{Z}[G]) = H_{4-s}(F;\mathbb{Z}) = 0$ for $s < 3$ and $H^3(G;\mathbb{Z}[G]) \cong H_1(F;\mathbb{Z})$ is a nontrivial finitely generated abelian group. Therefore $H^3(G;\mathbb{Z}[G]) \cong \mathbb{Z}$ [5].

Thus we may assume that F is an homology circle. Let $\tilde{G} = \pi/\nu'$ and let $t \in \tilde{G}$ represent a generator of the infinite cyclic group ν/ν'. Since F is finitely dominated a Wang sequence argument shows that $H_q(F';k)$ is a finitely generated $k[t,t^{-1}]$-module on which $t-1$ acts invertibly, for all $q > 0$. Then $H_q(F';\mathbb{F}_p)$ is finitely generated for all primes p and all $q > 0$. Since $H^s(\tilde{G};k[\tilde{G}]) = 0$ for all k and all $s < 4$, it follows that $H_q(F';\mathbb{F}_p) = 0$ for all primes p and all $q > 0$. Nontrivial finitely generated $\mathbb{Z}[t,t^{-1}]$-modules have nontrivial finite quotients, and so we may conclude that F' is acyclic.

Since M is a finite PD_4-space $C_*(\widetilde{M})$ is chain homotopy equivalent to a finite free $\mathbb{Z}[\pi]$-complex C_*. Thus $D_* = \mathbb{Z} \otimes_{\mathbb{Z}[\nu']} C_*$ is a finite free $\mathbb{Z}[\tilde{G}]$-complex, and is a resolution of $\mathbb{Z}$. Therefore $\tilde{G}$ is a PD_4-group. (In particular, we see again that $G = \tilde{G}/(\nu/\nu')$ is FP_∞.)

Since ν/ν' is a torsion-free abelian normal subgroup of $\tilde{G}$ the group ring $\mathbb{Z}[\tilde{G}]$ has a safe extension R, obtained by localising with respect to the nonzero elements of $\mathbb{Z}[t,t^{-1}]$. This means that R is a flat extension of $\mathbb{Z}[\tilde{G}]$ which is weakly finite and such that $R \otimes_{\mathbb{Z}[\tilde{G}]} \mathbb{Z} = 0$. (See page 23 of [7] and the references there.) Hence $R \otimes_{\mathbb{Z}[\tilde{G}]} D_*$ is a contractible complex of finitely generated free R-modules. It follows that $\chi(M) = \chi(R \otimes_{\mathbb{Z}[\tilde{G}]} D_*) = 0$. Since ν is an infinite FP_2 normal subgroup of π and π/ν has one end $\beta_1^{(2)}(\pi) = 0$ and $H^s(\pi;\mathbb{Z}[\pi]) = 0$ for $s \leq 2$. Therefore M is aspherical, by Theorem 3.5 of [7]. □

With this result we may now reformulate Theorem 3.9 of [7] as follows.

Corollary. *A finite PD_4-space M has a finitely dominated infinite regular covering space if and only if either M is aspherical, or has a 2-fold cover which is homotopy equivalent to the mapping torus of a self-homotopy equivalence of a PD_3-complex, or $\widetilde{M} \simeq S^2$ or S^3. If M is nonaspherical and has a finitely dominated regular covering space it is a PD_4-complex.*

Proof. Only the final sentence needs any comment. If $\widetilde{M} \simeq S^2$ or S^3 then $\pi_1(M)$ is virtually a PD_2-group or has two ends. In either case it is finitely presentable. This is also clear if M has a 2-fold cover which is the mapping torus of a self-homotopy equivalence of a PD_3-complex. Thus in all three cases M is a PD_4-complex. □

There are PD_n groups of type FF which are not finitely presentable, for each $n \geq 4$ [4]. The corresponding $K(G, 1)$ spaces are aspherical finite PD_n-spaces which are not PD_n-complexes.

Finiteness of M seems irrelevant to the conclusion of Theorem 5. (It is used here only in the calculation of $\chi(M)$.) Moreover the argument for Theorem 5 does not extend to the case when ν is an ascendant subgroup of π, as considered in [8] (where the FP_3 condition is also used). It would be of interest to find an argument along the following lines. Let C_* be a finite projective $\mathbb{Z}[\pi]$-complex with $H_0(C_*) \cong \mathbb{Z}$ and $H_1(C_*) = 0$. Show that $Hom_{\mathbb{Z}[\pi]}(H_2(C_*), \mathbb{Z}[\pi]) = 0$ if $[\pi : \nu] = \infty$ and $C_*|_\nu$ is chain homotopy equivalent to a finite projective $\mathbb{Z}[\nu]$-complex. The proofs of Theorem 3.9 of [7] and Theorem 6 of [8] would then apply, without needing to assume that ν is FP_3 or that M is finite.

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