FLAT 2-ORBIFOLD GROUPS

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Abstract. This is a summary of some of the basic facts about flat 2-orbifold groups, otherwise known as 2-dimensional crystallographic groups. We relate the geometric and topological presentations of these groups, and consider structures corresponding to decompositions of the orbifolds as fibrations or as unions.

An n-dimensional crystallographic group is a discrete subgroup π of the group $E(n) = \mathbb{R}^n \rtimes O(n)$ of isometries of euclidean n-space $\mathbb{R}^n$ which acts properly discontinuously on $\mathbb{R}^n$. The *translation subgroup* $\pi \cap \mathbb{R}^n$ is a lattice of rank n, with quotient a finite subgroup of $O(n)$, called the *holonomy group* of π. Since conjugation by π preserves the lattice, the holonomy group is conjugate in $GL(n, \mathbb{R})$ to a subgroup of $GL(n, \mathbb{Z})$. The quotient B of $\mathbb{R}^n$ by the action has a natural orbifold structure, recording the images of points with nontrivial stabilizers. The group π is then the *orbifold fundamental group* $\pi^{orb}(B)$.

It is well known that when $n = 2$ there are just 17 possibilities. We shall relate the presentations of the groups deriving from their structure as an extension of a finite group by a lattice to those deriving from the orbifold structure. We give explicit embeddings of each group in $E(2)$, where there is not an obvious choice. (However, we do not consider the issue of moduli, i.e., the parametrization of all such embeddings of a given flat 2-orbifold group.) The orbifold fibres over a 1-orbifold if and only if the group is an extension of $\mathbb{Z}$ or the infinite dihedral group D_∞. In the latter case the group is also a generalized free product with amalgamation (GFPA), corresponding to a decomposition of the orbifold along a codimension-1 suborbifold. Finally we describe the minimal proper covering relations between these orbifolds. All this material is well known; the only novelty here is perhaps in bringing this material together. We list several relevant expository articles at the end, although these are not explicitly invoked in the paper.

1. THE HOLONOMY GROUP

Let $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ and $R = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. If G is a nontrivial finite subgroup of $GL(2, \mathbb{Z})$ it is conjugate to one of the cyclic groups generated by A, $-I$, B, B^2, R or AR, or to a dihedral subgroup generated by $\{A, R\}$, $\{-I, R\}$, $\{-I, AR\}$, $\{B, R\}$, $\{B^2, R\}$ or $\{B^2, RB\}$. Let $\widetilde{\mathbb{Z}}^2$ denote $\mathbb{Z}^2$, considered as a G-module via the inclusion $G < GL(2;\mathbb{Z})$. The flat 2-orbifold groups are the extensions of such groups G by $\mathbb{Z}^2$, and so are determined by the cohomology group $H^2(G;\widetilde{\mathbb{Z}}^2)$.

In 10 of these 13 cases $H^2(G;\widetilde{\mathbb{Z}}^2) = 0$, and so the semidirect product $\mathbb{Z}^2 \rtimes G$ is the unique extension of G by $\mathbb{Z}^2$ (up to isomorphism). The eight semidirect products with $G \leq \langle A, R\rangle = GL(2,\mathbb{Z}) \cap O(2)$ embed as discrete cocompact subgroups of $E(2) = \mathbb{R}^2 \rtimes O(2)$ in the obvious way. The five other semidirect products embed in $Aff(2) = \mathbb{R}^2 \rtimes GL(2,\mathbb{R})$, and these embeddings may be conjugated into $E(2)$. We shall give explicit embeddings for each of the four non-split extensions.

2. PRESENTATIONS

We shall give presentations arising from the extensions and also those arising from the corresponding flat orbifold. Epimorphisms to D_∞ correspond to GFPA structures, arising naturally from Van Kampen's Theorem. These are essentially unique for $\mathbb{A}$, $\mathbb{M}b$ and Kb, since the kernel must contain the centre $\zeta\pi$. Each group is identified by the traditional crystallographic symbol (in square brackets) and by the now-standard orbifold symbol [4].

Let $\mathbb{I} = [[0, 1]]$ and $\mathbb{J} = [[0, 1]$ be the reflector interval and the interval with one reflector endpoint and one ordinary endpoint. Let Mb and $D(2, 2)$ be the Möbius band and the disc with two cone points, but with ordinary boundaries. Then $\mathbb{I}$ is the one-point union of two copies of $\mathbb{J}$, and so $\pi_1^{orb}(\mathbb{J}) = Z/2Z$ and $\pi_1^{orb}(\mathbb{I}) \cong \pi_1^{orb}(D(2, 2)) \cong D_\infty$.

The first four orbifolds ($T, \mathbb{A}, Kb$ and $\mathbb{M}b$) fibre over S^1.

Holonomy $G = 1$.

$[p1] = T$. $\quad \mathbb{Z}^2 = \langle x,\ y \mid xy = yx\rangle$.

In the subsequent presentations the generators a, b, c, d, j, n and r shall represent elements whose images in the holonomy group have matrices $A, B, B^2, AR, -I, BR$ and R, respectively, with respect to the basis $\{x, y\}$ for the translation subgroup $\mathbb{Z}^2$. (The other generators m, p, s, t, u, v, w, z do not have such fixed interpretations.)

Holonomy $Z/2Z = \langle AR\rangle$. In this case $H^2(G;\widetilde{\mathbb{Z}}^2) = Z/2Z$.

$[pm] = \mathbb{A} = S^1 \times \mathbb{I}$. $\quad \pi = \mathbb{Z} \times D_\infty \cong (\mathbb{Z} \oplus Z/2Z) *_{\mathbb{Z}} (\mathbb{Z} \oplus Z/2Z)$. This is the split extension.

$$\langle \mathbb{Z}^2,\ d \mid dx = xd,\ dyd = y^{-1},\ d^2 = 1\rangle.$$

Let $u = dy$. Then also

$$\langle d,\ u,\ x \mid dx = xd,\ ux = xu,\ d^2 = u^2 = 1\rangle.$$

The subgroups $\langle x+y, d\rangle$ and $\langle dx, y\rangle$ are isomorphic to $\pi^{orb}(\mathbb{M}b)$ and $\pi_1(Kb)$, respectively.

$[pg] = Kb = Mb \cup Mb$. $\quad \pi = \mathbb{Z} \rtimes_{-1} \mathbb{Z} \cong \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$.

$$\langle \mathbb{Z}^2,\ d \mid d^2 = x,\ dyd^{-1} = y^{-1}\rangle = \langle d,\ y \mid dyd^{-1} = y^{-1}\rangle.$$

Let $u = dy$. Then also

$$\langle d,\ u \mid d^2 = u^2\rangle.$$

The quotient of π by its centre $\langle x\rangle \cong \mathbb{Z}$ is D_∞, but this extension does not split.

We may embed π in $E(2)$ via $y \mapsto (\mathbf{j}, I_2)$ and $z \mapsto (\frac{1}{2}\mathbf{i}, \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right))$.

Holonomy $Z/2Z = \langle R\rangle$:

$[cm] = \mathbb{M}b = Mb \cup S^1 \times \mathbb{J}$. $\quad \pi \cong \mathbb{Z} *_{\mathbb{Z}} (\mathbb{Z} \oplus Z/2Z)$.

$$\langle \mathbb{Z}^2,\ r \mid rxr = y,\ r^2 = 1\rangle.$$

Let $z = xr$. Then also

$$\langle r,\ z \mid rz^2 = z^2 r,\ r^2 = 1\rangle.$$

Let τ be the involution of the normal subgroup $\langle xy^{-1}, r\rangle = \langle r, zrz^{-1}\rangle \cong D_\infty$ which swaps r and zrz^{-1}. Thus $\pi \cong D_\infty \rtimes_\tau \mathbb{Z}$. The centre is $\langle xy\rangle \cong \mathbb{Z}$ and $\pi/\zeta\pi \cong D_\infty$, but this extension does not split.

The subgroup $\langle xr, rx\rangle = \langle z, [r,z]\rangle$ is isomorphic to $\pi_1(Kb)$.

The next five orbifolds fibre over $\mathbb{I}$.

Holonomy $Z/2Z = \langle -I\rangle$:

$[p2] = S(2,2,2,2) = D(2,2) \cup D(2,2)$. $\quad \pi \cong D_\infty *_{\mathbb{Z}} D_\infty$.

$$\langle \mathbb{Z}^2,\ j \mid jxj = x^{-1},\ jyj = y^{-1},\ j^2 = 1\rangle.$$

Let $u = jx$ and $v = jy$. Then also

$$\langle j,\ u,\ v \mid j^2 = u^2 = v^2 = (juv)^2 = 1\rangle.$$

This group is a semidirect product of the normal subgroup $\langle ju\rangle$ with $\langle j, v\rangle$. Thus $\pi \cong \mathbb{Z} \rtimes D_\infty$.

Holonomy $D_4 = \langle -I, R\rangle$:

$[cmm] = \mathbb{D}(2, \overline{2}, \overline{2})$. $\quad \pi \cong \langle S(2,2,2,2),\ r \mid rxr = y,\ r^2 = (jr)^2 = 1\rangle$.

Let $u = jx$, $v = jy$ and $z = jr$. Then $v = rur$ and $juv = rzurur$. Then also

$$\langle r,\ u,\ z \mid r^2 = u^2 = z^2 = (rz)^2 = (zuru)^2 = 1\rangle.$$

This group is the semidirect product of the normal subgroup $\langle xy^{-1}, u\rangle$ with $\langle j, x\rangle$, and so $\pi \cong D_\infty \rtimes D_\infty$. The corresponding GFPA structure $\pi \cong (D_\infty \times Z/2Z) *_{D_\infty} D_\infty$ derives from the decomposition of the disc along a chord which separates the cone point from the corner points.

Holonomy $D_4 = \langle -I, AR\rangle$. In this case $H^2(G;\widetilde{\mathbb{Z}}^2) = (Z/2Z)^2$. The group π has a presentation

$$\langle \mathbb{Z}^2,\ d,\ j \mid dx = xd,\ dyd^{-1} = y^{-1},\ jxj = x^{-1},\ jyj = y^{-1},$$
$$(jd)^2 = y^e,\ d^2 = x^f,\ j^2 = 1\rangle.$$

We may assume that $0 \le e, f \le 1$. In all cases, $\langle x\rangle$ and $\langle y\rangle$ are the maximal infinite cyclic normal subgroups.

Two extension classes give isomorphic groups, and so there are three possibilities:

$[pmm] = \mathbb{D}(\overline{2},\overline{2},\overline{2},\overline{2})$. $\pi \cong (D_\infty \times Z/2Z) *_{D_\infty} (D_\infty \times Z/2Z)$.

This is the split extension (with $e = f = 0$). It is also $D_\infty \times D_\infty$:

$$\langle \mathbb{Z}^2,\ d,\ j \mid dx = xd,\ dyd = y^{-1},\ jxj = x^{-1},\ jyj = y^{-1},$$
$$d^2 = j^2 = (dj)^2 = 1\rangle.$$

Let $s = jdx$ and $t = dy$. Then also

$$\langle d,\ j,\ s,\ t \mid d^2 = j^2 = s^2 = t^2 = (st)^2 = (tj)^2 = (jd)^2 = (ds)^2 = 1\rangle,$$
$$\text{or}\quad \langle d,\ j,\ s,\ t \mid d, t \leftrightharpoons j, s,\ d^2 = j^2 = s^2 = t^2 = 1\rangle.$$

The GFPA structure derives from the decomposition of the disc along a chord which separates one pair of adjacent corner points from the others.

$[pmg] = \mathbb{D}(2,2) = S^1 \times \mathbb{J} \cup D(2,2)$. $\pi \cong (\mathbb{Z} \oplus Z/2Z) *_{\mathbb{Z}} D_\infty$.

This corresponds to $(e, f) = (1, 0)$. (The choice $(0, 1)$ gives an isomorphic group.)

$$\langle \mathbb{Z}^2,\ d,\ j \mid jxj = x^{-1},\ y = (jd)^2,\ dx = xd,\ d^2 = j^2 = 1\rangle.$$

Let $v = jx$. Then also

$$\langle d,\ j,\ v \mid djv = jvd,\ d^2 = j^2 = v^2 = 1\rangle.$$

The relation $djv = jvd$ is equivalent to $jdj = vdv$, since $j^2 = v^2 = 1$. Hence we also have $\pi \cong D_\infty *_{D_\infty} D_\infty$, This GFPA structure derives from the decomposition of the disc along a chord which separates the two cone points.

The subgroups $\langle jv\rangle$ and $\langle d, jdj\rangle$ are normal, and so $\pi \cong \mathbb{Z} \rtimes D_\infty$ and $\pi \cong D_\infty \rtimes D_\infty$.

We may embed π in $E(2)$ via $d \mapsto (-\frac{1}{2}\mathbf{j}, \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right))$, $j \mapsto (0, -I_2)$ and $v \mapsto (-\mathbf{i}, -I_2)$.

The index-2 subgroups of the group of $\mathbb{D}(2,2)$ corresponding to $\langle AR\rangle < D_4$ and $\langle -AR\rangle < D_4$ are isomorphic to $\pi_1(Kb) = \mathbb{Z} \rtimes_{-1} \mathbb{Z}$ and $\pi^{orb}(\mathbb{A}) = \mathbb{Z} \times D_\infty$, respectively.

$[pgg] = P(2,2) = Mb \cup D(2,2)$. $\pi \cong \mathbb{Z} *_{\mathbb{Z}} D_\infty$.

This corresponds to $e = f = 1$.

$\langle \mathbb{Z}^2,\ d,\ j \mid d^2 = x,\ (jd)^2 = y,\ jd^2j = d^{-2},\ d^2(jd)^2 = (jd)^2d^2,\ j^2 = 1\rangle.$

Let $v = jd^2$. Then this reduces to

$$\langle d,\ j,\ v \mid d^2 = jv,\ j^2 = v^2 = 1\rangle \quad \text{or just} \quad \langle d,\ j \mid (jd^2)^2 = j^2 = 1\rangle.$$

There is an automorphism which fixes j and swaps d and jd. We have $\pi/\langle (jd)^2\rangle \cong \pi/\langle d^2\rangle \cong D_\infty$, but these extensions do not split.

We may embed π in $E(2)$ via $d \mapsto (\frac{1}{2}\mathbf{i}, (\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}))$ and $j \mapsto (\frac{1}{2}(\mathbf{i}+\mathbf{j}), -I_2)$.

The index-2 subgroups of the group of $P(2,2)$ corresponding to $\langle AR\rangle < D_4$ and $\langle -AR\rangle < D_4$ are both isomorphic to $\pi_1(Kb)$.

The remaining eight orbifolds do not fibre over S^1 or $\mathbb{I}$.

Holonomy $Z/4Z = \langle A\rangle$:

$[p4] = S(2,4,4)$. $\quad \langle \mathbb{Z}^2,\ a \mid axa^{-1} = y^{-1},\ aya^{-1} = x,\ a^4 = 1\rangle.$

Let $u = a^2x$. Then also $\langle a, u \mid a^4 = u^2 = (au)^4 = 1\rangle$.

Holonomy $D_8 = \langle A, R\rangle$. In this case $H^2(G; \widetilde{\mathbb{Z}}^2) = (Z/2Z)$. The group π has a presentation

$\langle \mathbb{Z}^2,\ a,\ r \mid axa^{-1} = y^{-1},\ aya^{-1} = x,\ rxr = y,\ a^4 = r^2 = 1,\ (ar)^2 = y^e\rangle,$

where $e = 0$ or 1. Let $a = t^2x$. Then this reduces to

$$\langle a, r, v \mid ava = rva^2r,\ a^4 = r^2 = v^2 = (va)^4 = 1,\ (ar)^2 = r(t^2v)^e r\rangle.$$

$[p4m] = \mathbb{D}(\overline{2}, \overline{4}, \overline{4})$. This is the split extension. It reduces to

$$\langle a, r, v \mid rav = vra,\ a^4 = r^2 = v^2 = (ar)^2 = (va)^4 = 1\rangle,$$

since $a^2r = ra^2$. Let $w = au$ and $z = vra$. Then also

$$\langle r, w, z \mid r^2 = w^2 = z^2 = (rw)^4 = (wz)^2 = (zr)^4 = 1\rangle.$$

$[p4g] = \mathbb{D}(\overline{2}, 4)$. This is the non-split extension.

$$\langle a, r, v \mid (ar)^2 = ra^2vr,\ a^4 = r^2 = v^2 = (va)^4 = 1\rangle.$$

Since $va = (a^2r)a(a^2r)^{-1}$ and $ava^{-1} = a^{-1}rar$, this reduces to

$$\langle a, r \mid a^4 = r^2 = (a^{-1}rar)^2 = 1\rangle.$$

We may embed π in $E(2)$ via $a \mapsto (0, (\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}))$, $r \mapsto (\frac{1}{2}\mathbf{i}, (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}))$ and $v \mapsto (-\mathbf{i}, -I_2)$.

The index-2 subgroup of this group corresponding to $\langle -I, AR\rangle < D_8$ is the split extension. In particular, the groups with holonomy $Z/4Z$ or D_8 do not contain the groups of Kb, $\mathbb{D}(2,2)$ or $P(2,2)$.

Holonomy $Z/3Z = \langle B^2\rangle$:

$[p3] = S(3,3,3)$. $\quad \langle \mathbb{Z}^2, c \mid cxc^{-1} = x^{-1}y,\ cyc^{-1} = x^{-1},\ c^3 = 1\rangle.$

Let $u = cx$. Then also

$$\langle c, u \mid c^3 = u^3 = (cu)^3 = 1\rangle.$$

Holonomy $Z/6Z = \langle B\rangle$:

$[p6] = S(2,3,6)$. $\langle \mathbb{Z}^2,\ b \mid bxb^{-1} = y,\ byb^{-1} = x^{-1}y,\ b^6 = 1\rangle$.

Let $v = b^2x$. Then also

$$\langle b, v \mid b^6 = v^3 = (bv)^2 = 1\rangle.$$

Holonomy $D_6 = \langle B^2, R\rangle$:

$[p3m1] = \mathbb{D}(\overline{3},\overline{3},\overline{3})$. $\langle S(3,3,3),\ r \mid rxr = y,\ r^2 = (rc)^2 = 1\rangle$.

Let $s = rc$ and $t = crx$. Then also

$$\langle r, s, t \mid r^2 = s^2 = t^2 = (rs)^3 = (st)^3 = (tr)^3 = 1\rangle.$$

Holonomy $D_6 = \langle B^2, BR\rangle$:

$[p31m] = \mathbb{D}(3,\overline{3})$. $\langle S(3,3,3),\ n \mid nxn = x^{-1}y,\ ny = yn,\ n^2 = (nc)^2 = 1\rangle$.

Let $v = nx^{-1}y^{-1}$ and $w = nx^2y^{-1}$. Then also

$$\langle c, v, w \mid c^3 = v^2 = w^2 = 1,\ w = cvc^{-1}\rangle$$

$$\text{or} \quad \langle c, w \mid w^2 = c^3 = (c^{-1}wcw)^3 = 1\rangle.$$

Holonomy $D_{12} = \langle B, R\rangle$:

$[p6m] = \mathbb{D}(\overline{2},\overline{3},\overline{6})$. $\langle S(2,3,6),\ r \mid rxr = y,\ r^2 = (rb)^2 = 1\rangle$.

Let $m = brb^{-1} = b^2r$, $n = br$ and $p = rbxy^{-2}$. Then also

$$\langle m, n, p \mid m^2 = n^2 = p^2 = (mn)^6 = (np)^3 = (pm)^2 = 1\rangle.$$

The matrix B is not orthogonal. However conjugation by $\left(\begin{smallmatrix} -2 & 1 \\ 0 & \sqrt{3} \end{smallmatrix}\right)$) carries $\langle B, R\rangle$ into $O(2)$, and thus carries each of the subgroups $\mathbb{Z}^2 \rtimes G$ of $Aff(2)$ determined by the five groups with 3-torsion into $E(2)$.

3. Fibrations over $\mathbb{I}$

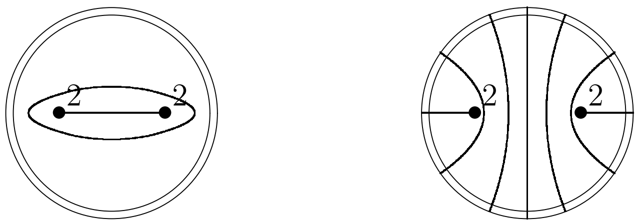
We shall say that two epimorphisms $\lambda, \lambda' : \pi \to D_\infty = \pi^{orb}(\mathbb{I})$ are equivalent if $\lambda' = \delta\lambda\alpha$ for some $\alpha \in Aut(\pi)$ and $\delta \in Aut(D_\infty)$. Such epis are most easily found by considering the possible kernels, which are maximal normal virtually-$\mathbb{Z}$ subgroups. (Note that since $Out(D_\infty) = Z/2Z$ and we are working with epimorphisms, it is sufficient to take δ to be either the automorphism which swaps the standard generators or the identity.)

All epimorphisms from $\pi^{orb}(S(2,2,2,2))$ to D_∞ are equivalent. These correspond to fibrations over $\mathbb{I}$ with general fibre S^1 and two singular fibres (reflector intervals connecting pairs of cone points).

All epimorphisms from $\pi^{orb}(\mathbb{D}(\overline{2},\overline{2},\overline{2},\overline{2}))$ to D_∞ are equivalent. (There are two maximal normal virtually-$\mathbb{Z}$ subgroups.) These correspond to the projections of $\mathbb{D}(\overline{2},\overline{2},\overline{2},\overline{2}) = \mathbb{I}\times\mathbb{I}$ onto its factors.

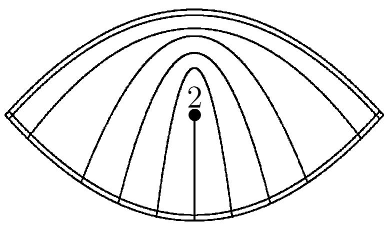
All epimorphisms from $\pi^{orb}(P(2,2))$ to D_∞ are equivalent. (There are two maximal normal virtually-$\mathbb{Z}$ subgroups.) These correspond to a fibration over $\mathbb{I}$ with general fibre S^1 and two singular fibres (the centreline of Mb, and one reflector interval connecting the cone points).

There are two equivalence classes of epimorphisms from $\pi^{orb}(\mathbb{D}(2,2))$ to D_∞. (There are two maximal normal virtually-$\mathbb{Z}$ subgroups, namely $\langle x\rangle$ and $\langle y\rangle$.) One corresponds to a fibration over $\mathbb{I}$ with general fibre S^1 and two singular fibres (one reflector interval connecting the cone points and the reflector curve). The other corresponds to a fibration with general fibre $\mathbb{I}$ and two singular fibres (two reflector intervals, each connecting a cone point to a reflector curve).



$\mathbb{D}(2,2)$

All epimorphisms from $\pi^{orb}(\mathbb{D}(2,\overline{2},\overline{2}))$ to D_∞ are equivalent. (There are two maximal normal virtually-$\mathbb{Z}$ subgroups.) These correspond to a fibration over $\mathbb{I}$ with general fibre $\mathbb{I}$ and one exceptional fibre (a reflector interval connecting the cone point to a reflector curve).



$\mathbb{D}(2,\overline{2},\overline{2})$

4. COVERINGS

If α and β are two flat orbifold groups and there is a monomorphism $\alpha\to\beta$ which is an isomorphism on the translation subgroups then we shall say that the corresponding orbifold cover is *equitranslational.*

In this case $\beta_1(\alpha) \geq \beta_1(\beta)$, the holonomy group G_α is conjugate in $GL(2,\mathbb{Z})$ to a subgroup of G_β, and the extension class $e_\beta \in H^2(G_\beta;\widetilde{\mathbb{Z}}^2)$ must restrict to e_α.

Such inclusions are easily determined from the lattices of subgroups of the two maximal subgroups $\langle A,R\rangle$ and $\langle B,R\rangle$ of $GL(2,\mathbb{Z})$ (modulo conjugacy). Determining the lattices of equitranslational covers of $\mathbb{D}(\overline{2},\overline{4},\overline{4})$ or of $\mathbb{D}(\overline{2},\overline{3},\overline{6})$ is straightforward, since in these cases the orbifold groups arising are split extensions. The only subtle point is how the nonsplit extensions (with holonomy a 2-group) restrict over subgroups of the holonomy groups. The index-2 subgroups of $\pi^{orb}(\mathbb{D}(\overline{2},4))$ corresponding to $\langle -I,AR\rangle$ are split extensions, and so $\mathbb{D}(2,2)$ and $P(2,2)$ do not cover $\mathbb{D}(\overline{2},4)$. The index-2 subgroups of $\pi^{orb}(P(2,2))$ corresponding to $\langle \pm AR\rangle$ are both $\pi_1(Kb)$, and so $\mathbb{A}$ does not cover $P(2,2)$.

The orientable orbifold $S(2,2,2,2)$ covers all flat orbifolds with holonomy of even order except for Kb, $\mathbb{A}$, $\mathbb{M}b$, $\mathbb{D}(3,\overline{3})$ and $\mathbb{D}(\overline{3},\overline{3},\overline{3})$. In particular, $\mathbb{D}(2,2)$, $\mathbb{D}(2,\overline{2},\overline{2})$ and $\mathbb{D}(\overline{2},\overline{2},\overline{2},\overline{2})$ are quotients by reflections across circles passing through 0, 2 or 4 cone points, respectively. The quotient by rotation of order 2 with both fixed points cone points is $S(2,4,4)$, while the quotient by rotation of order 3 with one fixed point a cone point is $S(2,3,6)$.

Similarly, $S(2,4,4)$ covers each of $\mathbb{D}(\overline{2},4)$ and $\mathbb{D}(\overline{2},\overline{4},\overline{4})$, and $S(3,3,3)$ covers all with holonomy divisible by 3.

If we drop the requirement that the translation subgroups coincide, there also less obvious inclusions. It remains necessary that G_α be conjugate in $GL(2,\mathbb{Q})$ to a subgroup of G_β. We may also use the (non)existence of reflector curves and/or corner points in testing whether one orbifold covers another. In all cases the subgroup generated by $2x$ and $2y$ is normal and of index 4 in the translation subgroup, and so the orbifolds have degree-4 self-coverings. (If $G \leq \langle A,R\rangle$ there is a degree-2 self-covering, since the subgroup generated by $x+y$ and $x-y$ is normal and of index 2 in the translation subgroup.)

For example, although AR and R are not conjugate in $GL(2,\mathbb{Z})$, they are conjugate in $GL(2,\mathbb{Z}[\frac{1}{2}])$. The inclusion $\mathbb{Z}\times D_\infty < D_\infty \rtimes_\tau \mathbb{Z}$ corresponds to the geometric fact that $\mathbb{A}$ covers $\mathbb{M}b$. Folding $\mathbb{M}b$ across its centerline gives $\mathbb{A}$, while the quotient of Kb by fibrewise reflection is $\mathbb{M}b$. However $P(2,2)$ is covered only by T, Kb, $S(2,2,2,2)$ and itself.

The involution $[x:y:z]\mapsto[x:-y:-z]$ of RP^2 has one fixed point and one fixed circle. Hence $P(2,2)$ covers $\mathbb{D}(2,2)$.

Rotating $\mathbb{D}(\overline{2},\overline{2},\overline{2},\overline{2})$ about its centre gives $\mathbb{D}(2,\overline{2},\overline{2})$. Conversely, folding $\mathbb{D}(2,\overline{2},\overline{2})$ across a diameter through the cone point and separating the corner points gives $\mathbb{D}(\overline{2},\overline{2},\overline{2},\overline{2})$. Folding $\mathbb{D}(2,2)$ across a diameter separating the cone points gives $\mathbb{D}(2,\overline{2},\overline{2})$.

Since $\mathbb{D}(2,2)$ has no corner points, it is not covered by $\mathbb{D}(2,\overline{2},\overline{2})$. Since $\mathbb{D}(\overline{2},4)$ has no corner points $\overline{4}$ with stabilizer D_8, it is not covered by $\mathbb{D}(\overline{2},\overline{4},\overline{4})$. Nor does it cover $\mathbb{D}(\overline{2},\overline{4},\overline{4})$.

The non-orientable orbifold $\mathbb{D}(\overline{2},\overline{3},\overline{6})$ is covered by all with holonomy divisible by 3. Rotating $\mathbb{D}(\overline{3},\overline{3},\overline{3})$ about its centre gives $\mathbb{D}(3,\overline{3})$. However $\mathbb{D}(3,\overline{3})$ does not cover $\mathbb{D}(\overline{3},\overline{3},\overline{3})$.

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