COMPLEMENTS OF CONNECTED HYPERSURFACES IN S^4

JONATHAN A. HILLMAN

To the memory of Tim Cochran

Abstract. Let X and Y be the complementary regions of a closed hypersurface M in $S^4 = X \cup_M Y$. We use the Massey product structure in $H^*(M;\mathbb{Z})$ to limit the possibilities for $\chi(X)$ and $\chi(Y)$. We show also that if $\pi_1(X) \neq 1$ then it may be modified by a 2-knot satellite construction, while if $\chi(X) \leq 1$ and $\pi_1(X)$ is abelian then $\beta_1(M) \leq 4$ or $\beta_1(M) = 6$. Finally we use TOP surgery to propose a characterization of the simplest embeddings of $F \times S^1$.

A closed hypersurface in S^n is orientable and has two complementary components, by the higher-dimensional analogue of the Jordan Curve Theorem. There have been sporadic papers presenting restrictions on the orientable 3-manifolds which may embed in S^4, but little is known about how many distinct embeddings there may be. (Here and in what follows, "embed" shall mean "embed as a TOP locally flat submanifold", unless otherwise qualified.) While the question of which rational homology 3-spheres embed smoothly in S^4 has received considerable attention, work on embeddings of more general 3-manifolds is very limited. Most of the relevant papers known to us are cited in [1].

The complementary components of embeddings of S^3 in S^4 are balls, by the Schoenflies Theorem. A result of Aitchison shows that every embedding of $S^2 \times S^1$ in S^4 has one complementary component homeomorphic to $S^2 \times D^2$ [18]. The other component is a 2-knot complement, with Euler characteristic $\chi = 0$ and fundamental group a 2-knot group, and so embeddings of $S^2 \times S^1$ in S^4 correspond to 2-knots. But for 3-manifolds M with $\beta = \beta_1(M) > 1$ even the possible Euler characteristics of the complementary components are not known.

In the first section we make some simple observations on the complementary components X and Y. We may assume that $1 - \beta \leq \chi(X) \leq 1 \leq \chi(Y) \leq 1 + \beta$. In §2 we use the Massey product structure in $H^*(M;\mathbb{Z})$ to show that if M fibres over an orientable base surface and the fibration has Euler number 1 then $\chi(X) = \chi(Y) = 1$ is the only possibility. At the other extreme, $\chi(X) = 1 - \beta$ is realizable only if the rational nilpotent completion of $\pi = \pi_1(M)$ is that of a free group. In the brief §3 we use a "satellite" construction based on 2-knots to modify the fundamental group of a complementary component which is not 1-connected, without changing the other complementary component. In §4 we show that $\pi_1(X)$ can be abelian only if $\beta \leq 4$ or $\beta = 6$, and give examples realizing these possibilities. In §5 we assume that M is Seifert fibred, with orientable base orbifold. If the generalized Euler invariant ε_S is 0 and $\chi(X) < 0$ then the regular fibre has nonzero image in $H_1(Y;\mathbb{Q})$, and so $\chi(X) > 1 - \beta$. If $\varepsilon_S \neq 0$ then $\chi(X) = \chi(Y) = 1$.

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When $M = F \times S^1$ or when M is the total space of an S^1-bundle with non-orientable base the simplest embeddings of M have one complementary component $X \simeq F$ and the other with cyclic fundamental group. In §6 we sketch how surgery may be used to identify such embeddings (up to s-cobordism). (No such argument is yet available when M fibres over an orientable base with Euler number 1.)

1. Euler characteristic and cup product

Let M be a closed connected orientable 3-manifold with fundamental group π, and let $\beta = \beta_1(M;\mathbb{Q})$. Let T_M be the torsion subgroup of $H_1(M;\mathbb{Z})$ and $\ell_M : T_M \times T_M \to \mathbb{Q}/\mathbb{Z}$ the torsion linking pairing.

Suppose M embeds in S^4, with complementary components X and Y. Let j_X and j_Y be the inclusions of M into X and Y, respectively. Then $\chi(X) + \chi(Y) = 2$.

Lemma 1. *Let $\gamma = \beta_1(X;\mathbb{Q})$. Then $\chi(X) = 1 + \beta - 2\gamma \equiv 1 + \beta$ mod (2), and $1 - \beta \leq \chi(X) \leq 1 + \beta$.*

Proof. The Mayer-Vietoris sequence for $S^4 = X \cup_M Y$ gives isomorphisms

$$H_i(M;\mathbb{Z}) \cong H_i(X;\mathbb{Z}) \oplus H_i(Y;\mathbb{Z}),$$

for $i = 1, 2$, while $H_j(X;\mathbb{Z}) = H_j(Y;\mathbb{Z}) = 0$ for $j > 2$. Moreover, $H_2(X;\mathbb{Z}) \cong H^1(Y;\mathbb{Z})$, by Poincaré-Lefshetz duality, and so $\beta_2(X) = \beta - \gamma$. Hence $\chi(X) = 1 + \beta - 2\gamma$, where $0 \leq \gamma \leq \beta$. □

We may assume X and Y are chosen so that $\chi(X) \leq \chi(Y)$. Thus if $\beta = 0$ then $\chi(X) = \chi(Y) = 1$, while if $\beta = 1$ then $\chi(X) = 0$ and $\chi(Y) = 2$.

Let T_X and T_Y be the torsion subgroups of $H_1(X;\mathbb{Z})$ and $H_1(Y;\mathbb{Z})$, respectively. Then $T_M \cong T_X \oplus T_Y$, and each of these summands is self-annihilating under ℓ_M, by Poincaré-Lefshetz duality. Hence ℓ_M is hyperbolic [12]. In particular, $T_Y \cong Ext(T_X, \mathbb{Z}) \cong Hom(T_X, \mathbb{Q}/\mathbb{Z})$, and so T_M is a direct double: it is (non-canonically) isomorphic to $T_X \oplus T_X$.

The cohomology ring $H^*(M;\mathbb{Z})$ is determined by the 3-fold product

$$\mu_M : \wedge^3 H^1(M;\mathbb{Z}) \to H^3(M;\mathbb{Z})$$

and Poincaré duality. Every finitely generated free abelian group H and linear homomorphism $\mu : \wedge^3 H \to \mathbb{Z}$ is realized by some closed orientable 3-manifold [20]. (If $\beta \leq 2$ then $\wedge^3 \mathbb{Z}^\beta = 0$, and so $\mu_M = 0$.)

Lemma 2. *The cup product 3-form μ_M is 0 if and only if all cup products of classes in $H^1(M;\mathbb{Z})$ are 0. Its restrictions to each of $\wedge^3 H^1(X;\mathbb{Z})$ and $\wedge^3 H^1(Y;\mathbb{Z})$ are 0.*

Proof. Poincaré duality implies immediately that $\mu_M = 0$ if and only if all cup products from $\wedge^2 H^1(M;\mathbb{Z})$ to $H^2(M;\mathbb{Z})$ are 0.

Since $H^3(X;\mathbb{Z}) = H^3(Y;\mathbb{Z}) = 0$, the restrictions of μ_M to $\wedge^3 H^1(X;\mathbb{Z})$ and $\wedge^3 H^1(Y;\mathbb{Z})$ are 0. □

See [15] for the parallel case of doubly sliced knots.

If $\mu_M \neq 0$ then $H^1(X;\mathbb{Z})$ and $H^1(Y;\mathbb{Z})$ must be nontrivial proper summands. In particular, no embedding of the 3-torus $S^1 \times S^1 \times S^1$ can have a complementary region Y with $H_1(Y;\mathbb{Z}) = 0$. However, if $\mu_M = 0$ this lemma places no constraint on the splitting $H^1(M;\mathbb{Z}) \cong H^1(X;\mathbb{Z}) \oplus H^1(Y;\mathbb{Z})$.

Any 3-manifold M may be obtained by 0-framed surgery on some r-component link L, with $r \geq \beta$. If $L = L_+ \cup L_-$ is the union of an s-component slice link

L_+ and an $(r-s)$-component slice link L_- then ambient surgery on S^3 in S^4 shows that M embeds in S^4, with complementary components having $\chi = 1 + 2s - r$ and $1 - 2s + r$. In particular, if L is a slice link then $\beta = r$ and there are embeddings realizing each value of $\chi(X)$ allowed by this lemma, including one with a 1-connected complementary region.

For instance, $\#^\beta(S^2 \times S^1)$ is the result of 0-framed surgery on the β-component trivial link, and so has embeddings realizing all the possibilities for Euler characteristics allowed by Lemma 1. In particular, it has an embedding with one complementary region $\natural^\beta(S^2 \times D^2)$, and the other having fundamental group $F(\beta)$. (In this case $\mu_M = 0$.)

The 3-torus is the result of 0-framed surgery on the Borromean rings $Bo = 6^3_2$. (We refer to the tables of [17]. This link shall play a role in the construction of other examples.) Let $T_g = \#^g T$ be the closed orientable surface of genus g. Then $T_g \times S^1$ is an iterated fibre sum of copies of $T \times S^1$, and so it may be obtained by 0-framed surgery on a $(2g+1)$-component link L which shares some of the Brunnian properties of Bo. It has an embedding as the boundary of $T_g \times D^2$, the regular neighbourhood of the unknotted embedding of T_g in S^4, with the other complementary region having fundamental group $\mathbb{Z}$. On the other hand, if $g \geq 1$ then $\mu_{T_g \times S^1} \neq 0$, and so no embedding has a complementary region Y with $\beta_1(Y) = 0$.

It is not hard to show that if $H \cong \mathbb{Z}^\beta$ with $\beta \leq 5$ then for every $\mu : \wedge^3 H \to \mathbb{Z}$ there is an epimorphism $\lambda : H \to \mathbb{Z}$ such that μ is 0 on the image of $\wedge^3 \mathrm{Ker}(\lambda)$. Hence there are splittings $H \cong A \oplus B$ with A of rank 3 or 4 such that μ restricts to 0 on each of $\wedge^3 A$ and $\wedge^3 B$. However if $\beta = 6$ this fails for the 3-form

$$\mu = e_1^* \wedge e_2^* \wedge e_3^* + e_1^* \wedge e_5^* \wedge e_6^* + e_2^* \wedge e_4^* \wedge e_5^*.$$

(Here $\{e_i^*\}$ is the basis for $Hom(\mathbb{Z}^6, \mathbb{Z})$ which is Kronecker dual to the standard basis $\{e_j\}$ of $\mathbb{Z}^6$.) For every epimorphism $\lambda : \mathbb{Z}^6 \to \mathbb{Z}$ there is a rank 3 direct summand A of $\mathrm{Ker}(\lambda)$ such that μ is nontrivial on $\wedge^3 A$. [This requires a little calculation. Suppose that $\lambda = \Sigma \lambda_i e_i^*$. If $\lambda_6 \neq 0$ then we may take A to be the direct summand containing $\langle f_1, f_2, f_3 \rangle$, where $f_j = \lambda_6 e_j - \lambda_j e_6$, for $1 \leq j \leq 3$, for then $\mu(f_1 \wedge f_2 \wedge f_3) = \lambda_6^3 \neq 0$. Similarly if λ_3 or λ_4 is nonzero. If $\lambda_3 = \lambda_4 = \lambda_6 = 0$ but $\lambda_1 \neq 0$ then we may take A to be the direct summand containing $\langle g_2, e_4, g_5 \rangle$, where $g_2 = \lambda_1 e_2 - \lambda_2 e_1$ and $g_5 = \lambda_1 e_5 - \lambda_5 e_1$. Similarly if λ_2 or λ_5 is nonzero.]

This example arose in a somewhat different context [3]. It is the cup product 3-form of the 3-manifold M given by 0-framed surgery on the 6-component link of Figure 6.1 of [3]. This link has certain "Brunnian" properties. All the 2-component sublinks, all but three of the 3-component sublinks and six of the 4-component sublinks are trivial. Thus M has embeddings in S^4 with $\chi(X) = -1$ or 1, corresponding to partitions of L into a pair of trivial sublinks, but there are no embeddings with $\chi(X) = -5$ or -3, since the condition on μ_M fails.

2. Massey products and lower central series

Massey product structures in the cohomology of M provide further obstructions. For instance, if $H^2(X; \mathbb{Q}) \cong \mathbb{Q}$ or 0 then all triple Massey products $\langle a, b, c \rangle$ of elements $a, b, c \in H^1(X; \mathbb{Q})$ are proportional.

Let $M(g; (1, e))$ be the total space of the S^1-bundle with base the closed orientable surface of genus g and Euler number $-e$. (This notation is consistent with that used for Seifert fibred 3-manifolds in §4 below.) Then $M = M(1; (1, 1))$ is the

$\mathbb{N}il^3$-manifold obtained by 0-framed surgery on the Whitehead link $Wh = 6^2_3$, and has fundamental group $\pi \cong F(2)/F(2)_{[3]}$. This group has a presentation

$$\pi = \langle x, y, z \mid z = xyx^{-1}y^{-1},\ xz = zx,\ yz = zy\rangle.$$

Every element of π has an unique normal form $x^m y^n z^p$. The images X, Y of x, y in $H_1(\pi;\mathbb{Z})/ \cong H_1(T;\mathbb{Z})$ form a (symplectic) basis. Let ξ, η be the Kronecker dual basis for $H^1(\pi;\mathbb{Z})$. Define functions ϕ_ξ, ϕ_η and $\theta : \pi \to \mathbb{Z}$ by

$$\phi_\xi(x^m y^n z^p) = \frac{m(1-m)}{2},\ \phi_\eta(x^m y^n z^p) = \frac{n(1-n)}{2} \text{ and } \theta(x^m y^n z^p) = -mn - p,$$

for all $x^m y^n z^p \in \pi$. (We consider these as inhomogeneous 1-cochains with values in the trivial π-module $\mathbb{Z}$.) Then

$$\delta\phi_\xi(g,h) = \xi(g)\xi(h), \quad \delta\phi_\eta(g,h) = \eta(g)\eta(h) \quad \text{and} \quad \delta\theta(g,h) = \xi(g)\eta(h),$$

for all $g, h \in \pi$. Thus $\xi^2 = \eta^2 = \xi \cup \eta = 0$, and the Massey triple products $\langle\xi,\xi,\eta\rangle$ and $\langle\xi,\eta,\eta\rangle$ are represented by the 2-cocycles $\phi_\xi\eta+\xi\theta$ and $\theta\eta+\xi\phi_\eta$, respectively. On restricting these to the subgroups generated by $\{x,z\}$ and $\{y,z\}$, we see that they are linearly independent. In fact, $\langle\xi,\xi,\eta\rangle\cup\eta$ and $\langle\xi,\eta,\eta\rangle\cup\xi$ each generate $H^3(\pi;\mathbb{Z})$ (i.e., these Massey products are the Poincaré duals of Y and X, respectively).

Since the components of Wh are unknotted M embeds in S^4, with $\chi(X) = \chi(Y) = 1$, and since $\beta = 2$ we have $\mu_M = 0$. On the other hand, M has no embedding with $\chi(X) = -1$, for otherwise $H^3(X;\mathbb{Z})$ would contain $\langle\xi,\xi,\eta\rangle \cup \eta$, and so be nontrivial.

A similar strategy may be used for $M = M(g;(1,1))$ and $\pi = \pi_1(M)$, when $g > 1$. Let $\{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g\}$ be the basis for $H = H^1(\pi;\mathbb{Z})$ which is Kronecker dual to a symplectic basis for $H_1(\pi;\mathbb{Z}) \cong H_1(F;\mathbb{Z})$. Then $H = A \oplus B$, where A and B are self-annihilating with respect to cup product on F. The Massey triple products $\langle\alpha_i,\alpha_i,\beta_i\rangle$ and $\langle\alpha_i,\beta_i,\beta_i\rangle$ (for $1 \le i \le g$) form a basis for $H^2(\pi;\mathbb{Z})$ which is Poincaré dual to the given basis for $H_1(\pi;\mathbb{Z})$. If $L \le H$ is a direct summand of rank $> g$ then there are $a \in L \cap A$ and $b \in L/A$ such that $a \cup b \ne 0$ in $H^2(F;\mathbb{Z})$. We may assume that $a = \alpha_1$ and then $b = \beta_1 + b'$, where b' is in the span of $\{\alpha_2, \beta_2, \dots, \alpha_g, \beta_g\}$. But then $\langle a,a,b\rangle \cup b \ne 0$. It follows that if $j : M \to S^4$ is any embedding then $H^1(X;\mathbb{Z})$ and $H^1(Y;\mathbb{Z})$ each have rank at most g, and so $\chi(X) = \chi(Y) = 1$. (See §5 for a 0-framed link representing M and giving rise to such an embedding.)

We shall let $G_{[n]}$ denote the nth term of the descending lower central series of a group G, defined inductively by $G_{[1]} = G$ and $G_{n+1]} = [G, G_{[n]}]$, for all $n \ge 1$. Similarly, the rational lower central series is given by letting $G^{\mathbb{Q}}_1 = G$ and $G^{\mathbb{Q}}_{k+1}$ be the preimage in G of the torsion subgroup of $G/[G, G^{\mathbb{Q}}_k]$. Then $G/G^{\mathbb{Q}}_k$ is a torsion free nilpotent group, and $\{G^{\mathbb{Q}}_k\}_{k\ge1}$ is the most rapidly descending series of subgroups of G with this property.

The 3-form μ_M is 0 if and only if $\pi/\pi^{\mathbb{Q}}_{[3]} \cong F(\beta)/F(\beta)^{\mathbb{Q}}_{[3]}$ [20]. However, this is a rather weak condition. The next lemma gives a stronger result.

Lemma 3. *If $H_1(Y;\mathbb{Z}) = 0$ then $\pi/\pi_{[k]} \cong F(\beta)/F(\beta)_{[k]}$, for all $k \ge 1$.*

Proof. If $H_1(Y;\mathbb{Z}) = 0$ then $H_2(X;\mathbb{Z}) = 0$, and T must be 0, by the non-degeneracy of ℓ_M, so $H_1(M;\mathbb{Z}) \cong H_1(X;\mathbb{Z}) \cong \mathbb{Z}^\beta$. Let $f : \vee^\beta S^1 \to X$ be any map such that $H_1(f;\mathbb{Z})$ is an isomorphism. Then j_X and f induce isomorphisms on all quotients of the lower central series, by Stallings' Theorem [19], and so $\pi/\pi_{[k]} \cong F(\beta)/F(\beta)_{[k]}$, for all $k \geq 1$. $\square$

If M is the result of surgery on a β-component slice link L then it has an embedding with a 1-connected complementary region, and so this lemma applies.

There are parallel results for the rational lower central series and the p-central series, for primes p, with coefficients $\mathbb{Q}$ and $\mathbb{F}_p$, respectively. In particular, if $\beta_1(Y) = 0$ then $\pi/\pi^{\mathbb{Q}}_{[k]} \cong F(\beta)/F(\beta)^{\mathbb{Q}}_{[k]}$, for all $k \geq 1$. These lower central series are dual to the Massey product structures for classes in $H^1(G;\mathbb{F})$, with $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F}_p$, and Stallings' Theorem can be refined to relate "freeness" of quotients of such series and the vanishing of higher Massey products [5]. In particular, the kernel of cup product from $\wedge^2 H^1(G;\mathbb{Q})$ to $H^2(G;\mathbb{Q})$ is isomorphic to $G^{\mathbb{Q}}_{[2]}/G^{\mathbb{Q}}_{[3]}$, by the argument of [20].

Unfortunately, the fact that $\mathrm{Ker}(\cup_X) \subseteq \mathrm{Ker}(\cup_M)$ does not have useful consequences for M. For if $\beta_1(X) < \beta$ then $\mathrm{Ker}(\cup_X)$ has rank at most $\binom{\beta_1(X)}{2} \leq \binom{\beta-1}{2} = \binom{\beta}{2} - \beta$, which is a lower bound for the rank of $\mathrm{Ker}(\cup_M)$. If $\beta_1(X) = \beta$ then $\beta_2(X) = 0$ so $\mu_M = 0$, and all cup products of degree-1 classes are 0.

3. KNOT SURGERY

We may modify embeddings by "knot surgery" on a complementary region, as follows. Let N_γ be a regular neighbourhood in X of a simple closed curve representing $\gamma \in \pi_1(X)$. Then $\overline{S^4 \setminus N_\gamma} \cong D^2 \times S^2$ contains Y and M. If K is a 2-knot with exterior $E(K)$ then $\Sigma = \overline{S^4 \setminus N_\gamma} \cup E(K)$ is a homotopy 4-sphere, and so is homeomorphic to S^4. The complementary components to M in Σ are $X_1 = \overline{X \setminus N_\gamma} \cup E(K)$ and $Y_1 = Y$. Let t be the image of a meridian for K in the knot group $\pi K = \pi_1(E(K))$. If γ has infinite order in $\pi_1(X)$ then $\pi_1(X_1) \cong \pi_1(X) *_{\mathbb{Z}} \pi K$; if it has finite order c then $\pi_1(X_1) \cong \pi_1(X) *_{Z/cZ} (\pi K/\langle\langle t^c \rangle\rangle)$.

When $M = S^2 \times S^1$ is embedded as the boundary of the trivial 2-knot, with $X = D^3 \times S^1$ and $Y = S^2 \times D^2$, the core $S^2 \times \{0\} \subset Y_1$ is K, realized as a satellite of the unknot in Σ. This "satellite" construction gives all possible embeddings of $S^2 \times S^1$ in S^4 (up to composition with self-homeomorphisms of domain and range), by Aithchison's result [18].

If $\gamma = 1$ then any simple closed curve representing γ is isotopic to one contained in a small ball, since homotopy implies isotopy for curves in 4-manifolds. Hence in this case the construction does not change the topology of X. If M embeds with one complementary component 1-connected and another embedding has a component with $H_1 = 0$ must that component also be 1-connected?

4. ABELIAN FUNDAMENTAL GROUP

In this section we shall show that manifolds with embeddings for which $\pi_1(X)$ is abelian are severely constrained.

Theorem 4. *Suppose M has an embedding in S^4 for which $\pi_1(X)$ is abelian. Then either $\beta \leq 4$ or $\beta = 6$. If $\beta = 0$ or 2 then $\pi_1(X) \cong Z/nZ$ or $\mathbb{Z} \oplus Z/nZ$, respectively, for some $n \geq 1$, while if $\beta = 1$, 3, 4 or 6 then $\pi_1(X) \cong \mathbb{Z}^r$, where $r = \lfloor \frac{\beta+1}{2} \rfloor$. If $\beta = 1$ or 3 then X is aspherical.*

Proof. Let $r = \beta_1(X)$, $A = \pi_1(X)$ and $\tau = T_X$. Then $2r \geq \beta$ and $A \cong \mathbb{Z}^r \oplus \tau$. Since A is abelian, $H_2(A;\mathbb{Z}) = A \wedge A \cong \mathbb{Z}^{\binom{r}{2}} \oplus \tau^r \oplus (\tau \wedge \tau)$. This is a quotient of $H_2(X;\mathbb{Z}) \cong \mathbb{Z}^{\beta - r}$, by Hopf's Theorem. Hence $\binom{r}{2} \leq \beta - r \leq r$, and so $r \leq 3$. If $\tau \neq 0$ then either $r = \beta = 0$ and $\tau \wedge \tau = 0$, or $r = 1$, $\beta = 2$ and $\tau \wedge \tau = 0$. In either case, τ is (finite) cyclic. If $\beta \neq 0$ or 2 then $\tau = 0$ and either $r = \beta = 1$, or $r = 2$ and $\beta = 3$ or 4, or $r = 3$ and $\beta = 6$.

Let $\Lambda_A = \mathbb{Z}[A]$. The chain complex of the universal cover $\widetilde{X}$ is chain homotopy equivalent to a finite complex C_* of projective Λ_A-modules, with $C_q = 0$ for $q > 3$, since X is a compact 4-manifold with nonempty boundary. Since $\pi_1(M)$ surjects onto $\pi_1(X) = H_1(X;\mathbb{Z})$ the boundary $\partial\widetilde{X}$ is connected, and so $H_i(\widetilde{X}, \partial\widetilde{X};\mathbb{Z}) = 0$ for $i \leq 1$. Therefore $H^q(X;\Lambda_A) = H^q(Hom_{\Lambda_A}(C_*, \Lambda_A)) = 0$ for $q > 2$, by Poincaré-Lefshetz duality. We shall show that if $r = \beta = 1$ or $r = 2$ and $\beta = 3$ then we may assume that $C_3 = 0$ also, and so $\Pi = H_2(C_*) \cong \pi_2(X)$ is the only potential obstruction to asphericity.

In each case, $\Lambda_A = \mathbb{Z}[\mathbb{Z}^r]$ is a noetherian domain for which all projective modules are free, and the alternating sum of the ranks of the modules C_q is $\chi(X) = 0$. If $r = \beta = 1$ then the submodule Z_1 of 1-cycles is free and $(Z_1 \to C_1 \to C_0)$ is a resolution of the augmentation module $H_0(\widetilde{X};\mathbb{Z}) = \mathbb{Z}$, by Schanuel's Lemma. Moreover, C_2 maps onto Z_1, since $H_1(C_*) = H_1(\widetilde{X};\mathbb{Z}) = 0$. Therefore C_* splits as

$$C_* \cong (C_3 \to Z_2) \oplus (Z_1 \to C_1 \to C_0),$$

and C_3 and Z_2 are free of the same rank. Now $\mathbb{Z} \otimes_\Lambda \Pi = 0$, since $H_q(X;\mathbb{Z}) = 0$ for $q \geq 2$. Therefore the differential $\partial_3 : C_3 \to Z_2$ is injective, and so $H_3(C_*) = 0$.

If $r = 2$ and $\beta = 3$ then $H_3(C_*) = H^1(X; \partial X; \Lambda_A) = 0$, since $H^0(\partial X; \Lambda_A) = 0$ and $\pi_1(X)$ has one end. In each case,

$$H_q(C_*) = H^q(Hom_{\Lambda_A}(C_*, \Lambda_A)) = 0 \quad \text{for } q \geq 3,$$

and so C_* is chain homotopy equivalent to a finite complex of free Λ_A-modules of length at most 2, by Wall's finiteness criterion [23]. Since $H_1(C_*) = 0$ and $\Sigma(-1)^q rank(C_q) = 0$ we see that $\Pi = 0$, so $H_q(\widetilde{X};\mathbb{Z}) = 0$ for $q \geq 1$. Thus X is aspherical. □

If $r = \beta = 0$ and $\tau = 0$ then X and Y are contractible. In the remaining cases X cannot be aspherical, since either $H_2(X;\mathbb{Z})$ is too big (if $\beta = 2$ or 4), or $H_3(X;\mathbb{Z})$ is too small (if $\beta = 6$).

Embeddings realizing these possibilities may be easily found. The simplest examples are for $\beta = 0, 1$ or 3, with $M \cong S^3$, $M = S^2 \times S^1$ or $T \times S^1 = S^1 \times S^1 \times S^1$ the boundary of a regular neighbourhood of a point or of the standard unknotted embedding of S^2 or T in S^4, respectively.

Other examples may be given in terms of representative links. When $\beta = 0$ the $(2, 2n)$ torus link gives examples with $X \cong Y$ and $\pi_1(X) \cong Z/nZ$. When $\beta = 1$ we may use any knot which bounds a slice disc $D \subset D^4$ such that $\pi_1(D^4 \setminus D) \cong \mathbb{Z}$, such as the unknot or the Kinoshita-Terasaka knot. (All such knots have Alexander polynomial 1. Conversely every Alexander polynomial 1 knot bounds a TOP locally flat slice disc with group $\mathbb{Z}$, by a striking result of Freedman.) The links 8^3_5 and 8^3_6 give further simple examples. (These each have a trivial 2-component sublink and an unknotted third component which represents a meridian of the first component or the product of meridians of the first two components, respectively.) When $\beta = 2$ any 2-component link with unknotted components and linking number 0, such as the trivial 2-component link or Wh, gives examples with $\pi_1(X) \cong \mathbb{Z}$. We may construct examples realizing $\mathbb{Z} \oplus Z/nZ$ by adjoining to Bo a fourth unknotted component which links only the first component, with linking number n. When $\beta = 3$ we may use the links Bo, 9^3_9 or 9^3_{18}. (These each have a trivial 2-component sublink and an unknotted third component which represents the commutator of the meridians of the first two components. However neither of the latter two links is Brunnian.)

Let L be the 4-component link obtained from Bo by adjoining a parallel to the third component, and let M be the 3-manifold M obtained by 0-framed surgery on L. Then the meridians of L represent a basis for $H_1(M;\mathbb{Z}) \cong \mathbb{Z}^4$, and $\mu_M = e_1^* \wedge e_2^* \wedge e_3^* + e_1^* \wedge e_2^* \wedge e_4^*$, where $\{e_i^*\}$ is the Kronecker dual basis. This link may be partitioned into the union of two trivial 2-component links in two essentially different ways, and ambient surgery gives two essentially different embeddings of M. If the sublinks are $\{L_1, L_2\}$ and $\{L_3, L_4\}$ then the complementary components have fundamental groups $\mathbb{Z}^2$ and $F(2)$. Otherwise, the complementary components are homeomorphic and have fundamental group $\mathbb{Z}^2$.

If M is an example with $\beta = 6$ and $\pi_1(X)$ and $\pi_1(Y)$ abelian then

$$\mu_M = e_1^* \wedge e_5^* \wedge e_6^* + e_2^* \wedge e_4^* \wedge e_6^* + e_3^* \wedge e_4^* \wedge e_5^* + e_1^* \wedge e_2^* \wedge \tilde{e}_6^* + e_1^* \wedge e_3^* \wedge \tilde{e}_5^* + e_2^* \wedge e_3^* \wedge \tilde{e}_4^*,$$

where $\{e_1^*, e_2^*, e_3^*\}$ is a basis for $H^1(X;\mathbb{Z})$ and $\{e_4^*, e_5^*, e_6^*\}$ and $\{\tilde{e}_4^*, \tilde{e}_5^*, \tilde{e}_6^*\}$ are bases for $H^1(Y;\mathbb{Z})$. The simplest link giving rise to such a 3-manifold is a 6-component link with all 2-component sublinks trivial, a partition into two trivial 3-component links, and also a partition into two copies of Bo. It also has some trivial 4-component sublinks, but no trivial 5-component sublinks. We shall not give further details.

In all of the above examples except for when $\beta = 2$ and $T_X \neq 0$ the group $\pi_1(Y)$ is also abelian. Note that Theorem 4 does *not* apply to $\pi_1(Y)$, as it uses the hypothesis $\beta_1(X) \geq \frac{1}{2}\beta$!

5. SEIFERT FIBRED 3-MANIFOLDS

We shall assume henceforth that M is Seifert fibred. Let $M = M(g;S)$ be the orientable Seifert fibred 3-manifold with base orbifold $T_g(\alpha_1, \dots, \alpha_r)$ and Seifert data $S = \{(\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)\}$, where $1 < \alpha_i$ and $(\alpha_i, \beta_i) = 1$, for all $1 \leq i \leq r$. If $c > 0$ we let also $M(-c;S)$ be the orientable Seifert fibred 3-manifold with base orbifold $\#^c RP^2(\alpha_1, \dots, \alpha_r)$ and Seifert data S. (Our notation is based on that of [10]. In particular, we do not assume that $0 < \beta_i < \alpha_i$.) If $r = 1$, we allow also the possibility $\alpha_1 = 1$. Let $\varepsilon_S = -\Sigma_{i=1}^{i=r}(\beta_i/\alpha_i)$ be the generalized Euler invariant of the Seifert bundle.

Let $p : M \to B$ be the projection to the base orbifold B, and let $|B|$ be the surface underlying B. If h is the image of the regular fibre in π then the subgroup generated by h is normal in π, and $\pi^{orb}(B) \cong \pi/\langle h \rangle$.

Lemma 5. *Let M a an orientable Seifert fibred 3-manifold. If B is nonorientable or if $\varepsilon_S \neq 0$ then $H^*(M;\mathbb{Q}) \cong H^*(\#^\beta S^2 \times S^1;\mathbb{Q})$. Otherwise, the image of h in $H_1(M;\mathbb{Q})$ is nonzero, and $H^*(M;\mathbb{Q}) \cong H^*(|B| \times S^1;\mathbb{Q})$.*

Proof. There is a finite regular covering $q : \widehat{M} \to M$, which is an S^1-bundle space with orientable base $\widehat{B}$, say. Let $G = Aut(q)$. Then $H^*(M;\mathbb{Q}) \cong H^*(\widehat{M};\mathbb{Q})^G$. If B is nonorientable or if $\varepsilon_S \neq 0$ then the regular fibre has image 0 in $H_1(M;\mathbb{Q})$, and so $H^*(\widehat{B};\mathbb{Q})$ maps onto $H^*(M;\mathbb{Q})$. Hence all cup products of degree-1 classes are 0. In such cases, $H^*(M;\mathbb{Q})\cong H^*(\#^\beta S^2\times S^1;\mathbb{Q})$. Otherwise, $\widehat{M}\cong\widehat{B}\times S^1$ and G acts orientably on each of S^1 and $\widehat{B}$. Hence the image of h in $H_1(M;\mathbb{Q})$ is nonzero and $H^*(M;\mathbb{Q})\cong H^*(|B|\times S^1;\mathbb{Q})$. □

We may use the observations on cup product from §1 to extract some information on the image of the regular fibre under the maps $H_1(j_X)$ and $H_1(j_Y)$.

Theorem 6. *Let $M=M(g;S)$ where $g\geq 1$ and $\varepsilon_S=0$. If M embeds in S^4 then $\chi(X)>1-\beta=-2g$ and $\chi(Y)<1+\beta=2g+2$. If $\chi(X)<0$ then the image of h in $H_1(Y;\mathbb{Q})$ is nontrivial.*

Proof. Let $\{a_i^*,b_i^*;1\leq i\leq g\}$ be the images in $H^1(M;\mathbb{Q})$ of a symplectic basis for $H^1(|B|;\mathbb{Q})$. Then $a_i^*(h)=b_i^*(h)=0$ for all i. Let $\theta\in H^1(M;\mathbb{Q})$ be such that $\theta(h)\neq 0$. By Lemma 5 we have

$$H^*(M;\mathbb{Q})\cong H^*(|B|\times S^1;\mathbb{Q})\cong\mathbb{Q}[\theta,a_i^*,b_i^*,\ \forall\ i\leq g]/I,$$

where I is the ideal $(\theta^2,a_i^{*2},b_i^{*2},\theta a_i^*b_i^*-\theta a_j^*b_j^*,a_i^*a_j^*,b_i^*b_j^*,\ \forall\ 1\leq i<j\leq g)$.

Since $\theta a_1^*b_1^*\neq 0$ the triple product $\mu_M\neq 0$, and so M has no embedding with $\beta_2(Y)=0$ (see §1). Hence $\chi(X)=1-\beta$ ($\Leftrightarrow\chi(Y)\neq 1+\beta$) is impossible.

If $\chi(X)<0$ then $\beta_1(X)>g+1$, and so the image of $H^1(X;\mathbb{Q})$ in $H^1(M;\mathbb{Q})$ must contain some pair of classes from the image of $H^1(|B|;\mathbb{Q})$ with nonzero product. But then it cannot also contain θ, since all triple products of classes in $H^1(X;\mathbb{Q})$ are 0. Thus the image of $H^1(Y;\mathbb{Q})$ must contain a class which is nontrivial on h, and so $j_Y(h)\neq 0$ in $H_1(Y;\mathbb{Q})$. □

In particular, if $g=1$ then $\chi(X)=0$ and $\chi(Y)=2$.

Theorem 6 also follows from Lemma 3, since the centre of π is not contained in the commutator subgroup $\pi_{[2]}=[\pi,\pi]$.

If the base orbifold B is nonorientable or if $\varepsilon_S\neq 0$ then $\mu_M=0$, by Lemma 5, and so the argument of Theorem 6 does not extend to these cases. However, Lemma 5 also suggests that when $\varepsilon_S\neq 0$ we should be able to use Massey product arguments as in §2 (where we considered the case $S=\emptyset$).

Theorem 7. *Let $M=M(g;S)$, where $\varepsilon_S\neq 0$. If M embeds in S^4 with complementary regions X and Y then $\chi(X)=\chi(Y)=1$.*

Proof. The group $\pi=\pi_1(M(g;S))$ has a presentation

$$\langle x_1,y_1,\dots,x_g,y_g,c_1,\dots,c_r,h\mid\Pi[a_i,b_i]\Pi c_j=1,\ c_i^{\alpha_i}h^{\beta_i}=1,\ h\ central\rangle.$$

We may assume that $g\geq 1$, for if $g=0$ then M is a $\mathbb{Q}$-homology 3-sphere and the result is clear. To calculate cup products and Massey products of pairs of elements of a standard basis for $H^1(\pi;\mathbb{Q})$ (corresponding to the Kronecker dual of a symplectic basis for $H_1(|B|;\mathbb{Q})$), it suffices to reduce to the case $g=1$. Let $G=\pi/\langle\langle x_2,y_2,\dots,x_g,y_g\rangle\rangle$, so G has a presentation

$$\langle x,y,c_1,\dots,c_r,h\mid[x,y]\Pi c_j=1,\ c_i^{\alpha_i}h^{\beta_i}=1,\ h\ central\rangle.$$

Let $G_\tau=\langle\langle c_1,\dots,c_r,h\rangle\rangle$, and let H be the preimage in G of the torsion subgroup of $G/[G,G_\tau]$. Then $G_\tau/H\cong\mathbb{Z}$, with generator t, say, and $[x,y]=t^e$ for some

$e \neq 0$. Every element has a normal form $g = x^m y^n t^p w$, with $w \in H$. Define functions ϕ_ξ, ϕ_η and $\theta : \pi \to \mathbb{Q}$ by

$$\phi_\xi(x^m y^n t^p w) = \frac{m(1-m)}{2}, \quad \phi_\eta(x^m y^n t^p w) = \frac{n(1-n)}{2}$$

$$\text{and} \quad \theta(x^m y^n t^p w) = -mn - \frac{p}{e},$$

for all $x^m y^n t^p w \in G$. (In effect, we are passing to the $\mathbb{N}il^3$-group G/H, with presentation $\langle x, y, t \mid [x, y] = t^e,\ t\ \mathit{central}\rangle$.) We may now complete the argument as in §2, and we may conclude that only $\chi(X) = \chi(Y) = 1$ is possible when $\varepsilon_S \neq 0$. □

If $\chi(X) = 0$ and h has nonzero image in $H_1(X; \mathbb{Q})$ then S is skew-symmetric (i.e., the Seifert data occurs in pairs $\{(a, b), (a, -b)\}$), by the main result of [8]. (In particular, this must be the case if $g = 0$.) Conversely, if S is skew-symmetric and all cone point orders a_i are odd then $M(0; S)$ embeds smoothly. Since $\beta = 1$ we must have $\chi(X) = 0$ and $H_1(Y; \mathbb{Q}) = 0$. (In fact, for the embedding constructed on page 693 of [2] the component X has a fixed point free S^1-action.) Hence also $M(g; S)$ embeds smoothly (as in Lemma 3.3 of [2]).

If ℓ_M is hyperbolic then all even cone point orders have the same 2-adic valuation, by Theorem 3.7 of [2] (when $g < 0$) and Lemma 6 of [9] (when $g \geq 0$).

Donald has stronger results for the case of smooth embeddings, using gauge theoretic methods rather than algebraic topology [4]. If $M(g; S)$ embeds smoothly and $\varepsilon_S = 0$ then S is skew-symmetric. (Thus if $\varepsilon_S = 0$ and all cone point orders are odd then $M(g; S)$ embeds smoothly if and only if S is skew-symmetric.) If $M(-c; S)$ (with $c > 0$) embeds smoothly then S is weakly skew-symmetric (i.e., the data occurs in pairs $\{(a, b), (a, -b')\}$, where $b' = b$ or $bb' \equiv 1 \bmod (a)$) and all even cone point orders are equal.

Are there further obstructions related to 2-torsion in the cone point orders of the base orbifolds B? What are the possible values of $\chi(X)$ for embeddings of $M(g; S)$ (with $\varepsilon_S = 0$) or $M(-c; S)$?

6. RECOGNIZING THE SIMPLEST EMBEDDINGS

The simplest 3-manifolds to consider in the present context are perhaps the total spaces of S^1-bundles over surfaces. Most of those which embed have canonical "simplest" embeddings. We give some evidence that these may be characterized by the conditions $\pi_1(X) \cong \pi_1(F)$, where F is the base, and $\pi_1(Y)$ is abelian.

If $p : E \to F$ is an S^1-bundle with base a closed surface F and orientable total space E then $\pi_1(F)$ acts on the fibre via $w = w_1(F)$, and such bundles are classified by an Euler class $e(p)$ in $H^2(F; \mathbb{Z}^w) \cong \mathbb{Z}$. If we fix a generator $[F]$ for $H_2(F; \mathbb{Z}^w)$ we may define the Euler number of the bundle by $e = e(p)([F])$. (We may change the sign of e by reversing the orientation of E.) Let h be the image of the fibre in $\pi = \pi_1(E)$.

Suppose first that $F \cong T_g$. Then $E \cong M(g; (1, e))$ can only embed in S^4 if $e = 0$ or ± 1, since $T_E = 0$ if $e = 0$ and is cyclic of order e otherwise. If $e = 0$ then $E \cong T_g \times S^1$. There is a canonical embedding $j_g : T_g \times S^1 \to S^4$, as the boundary of a regular neighbourhood of the standard smooth embedding $T_g \subset S^3 \subset S^4$. Let X_g and Y_g be the complementary components. Then $X_g \cong T_g \times D^2$ and $Y_g \simeq S^1 \vee \bigvee^{2g} S^2$, and so $\pi_1(Y_g) \cong \mathbb{Z}$.

We shall assume henceforth that $g \geq 1$, since embeddings of $S^2 \times S^1$ and $S^3 = M(0;(1,1))$ may be considered well understood.

Lemma 8. *Let $j : T_g \times S^1 \to S^4$ be an embedding such that $\pi_1(X) \cong \pi_1(T_g)$. Then X is s-cobordant rel ∂ to $X_g = T_g \times D^2$.*

Proof. Let $\widetilde{X}$ be the universal cover, with boundary $\partial\widetilde{X} \cong T_g \times \mathbb{R}$, and let $\Gamma = \mathbb{Z}[\pi_1(F)]$. Then $H_q(\widetilde{X};\mathbb{Z}) = 0$ and $H^q(X;\Gamma)) = H_{4-q}(X, \partial X;\Gamma) = H_{4-q}(\widetilde{X}, T_g;\mathbb{Z}) = 0$ for $q > 2$, by Poincaré-Lefshetz duality and the long exact sequence of the pair $(X, \partial X)$. Therefore the equivariant chain complex for $\widetilde{X}$ is chain homotopy equivalent to a complex P_* of finitely generated projective Γ-modules which is of length 2, by Wall's finiteness criteria [23]. Hence there is an exact sequence

$$0 \to \Pi \to P_2 \to P_1 \to P_0 \to \mathbb{Z} \to 0,$$

where $\Pi = \pi_2(X)$. Hence Π is a finitely generated projective Γ-module, by Schanuel's Lemma (and the fact that $c.d.\pi_1(T_g) = 2$.) Since $H_3(\pi_1(T_g);\mathbb{Z}) = 0$, the Cartan-Leray spectral sequence of the universal cover gives a short exact sequence

$$0 \to \mathbb{Z} \otimes_\Gamma \Pi \to H_2(X;\mathbb{Z}) \to H_2(\pi_1(T_g);\mathbb{Z}) \to 0.$$

Now $H_2(X;\mathbb{Z}) \cong H_2(\pi_1(T_g);\mathbb{Z}) \cong \mathbb{Z}$, and so $\mathbb{Z} \otimes_\Gamma \Pi = 0$. Since $\pi_1(T_g)$ satisfies the weak Bass Conjecture, it follows that $\Pi = 0$ [6]. Hence $H_q(\widetilde{X};\mathbb{Z}) = 0$ for all $q \geq 1$, and so X is aspherical. Any homeomorphism from ∂X to ∂X_g which preserves the product structure extends to a homotopy equivalence of pairs $(X, \partial X) \simeq (X_g, \partial X_g)$. Now $L_5(\pi_1(T_g))$ acts trivially on the s-cobordism structure set $S_{TOP}(X_g, \partial X_g)$, by Theorem 6.7 and Lemma 6.9 of [7]. Therefore X and X_g are TOP s-cobordant (rel ∂). □

If $\pi_1(Y) \cong \mathbb{Z}$ then $\Sigma = Y \cup (T_g \times D^2)$ is 1-connected, since $\pi_1(Y)$ is generated by the image of h, and $\chi(\Sigma) = 2$. Hence Σ is a homotopy 4-sphere, containing a locally flat copy of T_g with exterior Y.

Lemma 9. *If there is a map $f : Y \to Y_g$ which extends a homeomorphism of the boundaries then Y is homeomorphic to Y_g.*

Proof. Let $\Lambda = \mathbb{Z}[t, t^{-1}]$ be the group ring of $\pi_1(Y) = \langle t \rangle$.

We see easily that $H_q(Y;\Lambda) = H^q(Y;\Lambda) = 0$ for $q > 2$, by Poincaré-Lefshetz duality (and using the fact that $\partial Y = T_g \times S^1$). As in Lemma 8 it follows that the equivariant chain complex for $\widetilde{Y}$ is chain homotopy equivalent to a finite projective Λ-complex Q_* of length 2, and so there is an exact sequence

$$0 \to \Pi \to Q_2 \to Q_1 \to Q_0 \to \mathbb{Z} \to 0,$$

where $\Pi = \pi_2(Y)$. All projective Λ-modules are free, and the alternating sum of the ranks of the modules Q_i is $\chi(Y) = 2g$. Applying Schanuel's Lemma to this resolution of $\mathbb{Z}$ and to the standard short exact sequence

$$0 \to \Lambda \to \Lambda \to \mathbb{Z} \to 0,$$

we see that $\Pi \cong \Lambda^{2g}$. In particular, this holds also for Y_g.

If $f : Y \to Y_g$ restricts to a homeomorphism of the boundaries then $\pi_1(f)$ is an isomorphism. Comparison of the long exact sequences of the pairs shows that f induces an isomorphism $H_4(Y, \partial Y;\mathbb{Z}) \cong H_4(Y, \partial Y;\mathbb{Z})$, and so has degree 1. Therefore $\pi_2(f) = H_2(f;\Lambda)$ is onto, by Poincaré-Lefshetz duality. Since $\pi_2(Y)$ and

$\pi_2(Y_g)$ are each free of rank $2g$, it follows that $\pi_2(f)$ is an isomorphism, and so f is a homotopy equivalence, by the Whitehead and Hurewicz Theorems.

Thus f is a homotopy equivalence *rel* ∂, by the HEP, and so it determines an element of the structure set $S_{TOP}(Y_g, \partial Y_g)$. The group $L_5(\mathbb{Z})$ acts trivially on the structure set, as before, and so the normal invariant gives a bjection $S_{TOP}(Y_g, \partial Y_g) \cong H^2(Y_g, \partial Y_g; \mathbb{F}_2) \cong H_2(Y_g; \mathbb{F}_2)$. Since $H_2(\mathbb{Z}; \mathbb{F}_2) = 0$ the Hurewicz homomorphism maps $\pi_2(Y_g)$ onto $H_2(Y_g; \mathbb{F}_2)$. Therefore there is an $\alpha \in \pi_2(Y_g)$ whose image in $H_2(Y_g; \mathbb{F}_2)$ is the Poincaré dual of the normal invariant of f. Let f_α be the composite of the map from Y_g to $Y_g \vee S^4$ which collapses the boundary of a 4-disc in the interior of Y_g with $id_{Y_g} \vee \alpha\eta^2$, where η^2 is the generator of $\pi_4(S^2)$. Then f_α is a self homotopy equivalence of $(Y_g, \partial Y_g)$ whose normal invariant agrees with that of f. (See Theorem 16.6 of [22].) Therefore f is homotopic to a homeomorphism $Y \cong Y_g$. ∎

However, finding such a map f to begin with seems difficult. Can we somehow use the fact that Y and Y_g are subsets of S^4? In fact, Y must be homeomorphic to Y_g if $g \geq 3$, according to [13].

Suppose now that W is an s-cobordism *rel* ∂ from X to $X_g = T_g \times D^2$, and that $Y \cong Y_g$. Since $g \geq 1$ the 3-manifold $T_g \times S^1$ is irreducible and sufficiently large. Therefore $\pi_0(Homeo(T_g \times S^1)) \cong Out(\pi)$ [21]. If $g > 1$ then $\pi_1(T_g)$ has trivial centre, and so $Out(\pi) \cong \left(\begin{smallmatrix} Out(\pi_1(T_g)) & 0 \\ \mathbb{Z}^{2g} & \mathbb{Z}^\times \end{smallmatrix}\right)$. It follows easily that every self homeomorphism of $T_g \times S^1$ extends to a self homeomorphism of $F \times D^2$. Attaching $Y \times [0,1] \cong Y_g \times [0,1]$ to W along $T_g \times S^1 \times [0,1]$ gives an s-concordance from j to j_g (i.e, one whose complementary regions are s-cobordisms *rel* ∂).

If $g = 1$ then $X \cong T \times D^2$ and $Out(\pi) \cong GL(3, \mathbb{Z})$. Automorphisms of π are generated by those which may be realized by homeomorphisms of $T \times D^2$ together with those that may be realized by homeomorphisms of Y_1 [16]. Thus if embeddings of T with group $\mathbb{Z}$ are standard so are embeddings of $S^1 \times S^1 \times S^1$ with both complementary components having abelian fundamental groups.

The situation is less clear for bundles over T_g with Euler number ± 1. We may construct embeddings of such manifolds by fibre sum of an embedding of $T_g \times S^1$ with the Hopf bundle $\eta : S^3 \to S^2$. However, it is not clear how the complements change under this operation. There are natural 0-framed links representing such bundle spaces. As we saw earlier, $M(1; (1,1))$ may be obtained by 0-framed surgery on the Whitehead link. This is an interchangeable 2-component link, and so $M(1; (1,1))$ has an embedding with $X \cong Y \simeq S^1 \vee S^2$ and $\pi_1(X) \cong \pi_1(Y) \cong \mathbb{Z}$. Is this embedding characterized by these conditions? (Once again, it is enough to find a map which restricts to a homeomorphism on boundaries.)

The product $M(1; (1,0)) \cong S^1 \times S^1 \times S^1$ may be obtained by 0-framed surgery on the Borromean rings. Changing the framing on one component of Bo to 1, and applying a Kirby move to isolate this component gives the disjoint union of Wh and the unknot. Since the linking numbers are 0 the framings are unchanged, and we may delete the isolated 1-framed unknot. The corresponding modification of the standard 0-framed $(2g+1)$-component link L representing $T_g \times S^1$ involves changing the framing of the component L_{2g+1} whose meridian represents the central factor of π. Performing a Kirby move and deleting an isolated 1-framed unknot gives a 0-framed $2g$-component link representing $M(g; (1,1))$. Since the original link had partitions into two trivial links with $g + 1$ and g components respectively, the new link has a partition into two trivial g-component links. However this is the only partition into slice sublinks, for as we saw in §2 consideration of the Massey product structure shows that all embeddings of $M(g;(1,1))$ have $\chi(X)=\chi(Y)=1$.

Suppose now that F is nonorientable. Then $F\cong\#^cRP^2$, where $c=2-\chi(F)\geq 1$, and $M(-c;(1,e))$ embeds if and only if it embeds as the boundary of a regular neighbourhood of an embedding of F with normal Euler number e. We must have $e\leq 2c$ and $e\equiv 2c$ *mod* (4) [2]. The standard embedding of RP^2 in S^4 is determined up to composition with a reflection of S^4. The complementary regions are each homeomorphic to a disc bundle over RP^2 with normal Euler number 2, and so have fundamental group $Z/2Z$. The standard embeddings of $\#^cRP^2$ are obtained by taking iterated connected sums of these building blocks $\pm(S^4,RP^2)$, and in each case the exterior has fundamental group $Z/2Z$. The regular neighbourhoods of $\#^cRP^2$ are disc bundles with boundary $M(-c;(1,e))$. Thus $M(-c;(1,e))$ has a standard embedding with one complementary component $X_{c,e}$ a disc bundle over $\#^cRP^2$ and the other component $Y_{c,e}$ having fundamental group $Z/2Z$.

The constructions in the appendix to [2] suggest framed link presentations for $M(-c;(1,e))$. The standard embedding corresponds to a 0-framed $(c+1)$-component link assembled from copies of the $(2,4)$-torus link 4^2_1 and its reflection. This is the union of an unknot and a trivial c-component link, but has no other partitions into slice links. However, we can do better if we recall that $\#^cRP^2\cong(\#^{c-2g}RP^2)\#T_g$ for any g such that $2g<c$. Using copies of $\pm 4^2_1$ and Bo accordingly, for each $e\leq 2c$ such that $e\equiv 2c$ *mod* (4) we find a representative link with partitions into trivial sublinks corresponding to all the values $\chi(X)\geq 2-\frac{|e|}{2}$. (Note Figure A.3 of [2].) Are any other values realized?

We may again argue that if j is an embedding of $M(-c;(1,e))$, where $c\geq 2$, and $\pi_1(X)\cong\pi_1(\#^cRP^2)$ then X is aspherical, and hence is s-cobordant to X_e. Moreover, if $\pi_1(Y)=Z/2Z$ then Y is the exterior of an embedding of $\#^cRP^2$ in S^4 with normal Euler number e. Kreck has shown that in certain cases embeddings of $\#^cRP^2$ with group $Z/2Z$ must be standard, and we should again expect that j is s-concordant to a standard embedding [14]. In particular, Kreck's result includes the case when $F=Kb$ (i.e., $c=2$). Hence embeddings of the half-turn flat 3-manifold $M(-2;(1,0))$ and of the $\mathbb{N}il^3$-manifold $M(-2;(1,4))$ with $\pi_1(X)\cong\pi_1(Kb)$ and $\pi_1(Y)=Z/2Z$ are standard.

Seven of the thirteen 3-manifolds with elementary amenable fundamental groups that embed are total spaces of S^1-bundles (namely, S^3, S^3/Q, $S^2\times S^1$, $S^1\times S^1\times S^1$, $M(-2;(1,0))$, $M(1;(1,1)$ and $M(-2;(1,4))$). Two of these and five of the others are the result of surgery on 2-component links with trivial component knots. (See [2].) The thirteenth such 3-manifold is the Poincaré homology sphere S^3/I^*, which bounds a contractible TOP 4-manifold C (as do all homology 3-spheres) and so embeds in the double $DC\cong S^4$. However, it is well known that S^3/I^* does not embed smoothly.

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School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia
E-mail address: `jonathan.hillman@sydney.edu.au`