Cantor-winning sets and their applications

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July 6, 2017

Abstract

We introduce and develop a class of *Cantor-winning* sets that share the same amenable properties as the classical winning sets associated to Schmidt's (α, β)-game: these include maximal Hausdorff dimension, invariance under countable intersections with other Cantor-winning sets and invariance under bi-Lipschitz homeomorphisms. It is then demonstrated that a wide variety of badly approximable sets appearing naturally in the theory of Diophantine approximation fit nicely into our broad-reaching framework. As applications of this phenomenon we answer several previously open questions, including some related to the Mixed Littlewood conjecture and approximation by multiplicative semigroups of integers.

1 Introduction

1.1 Badly approximable sets

The set **Bad** of badly approximable real numbers plays an important role in the theory of Diophantine approximation. Recall, a real number x is called *badly approximable* if there exists a constant $c(x) > 0$ such that $|x - p/q| \geqslant c(x)/q^2$ for all rational numbers p/q. It is well known that the set of all badly approximable numbers is very small in the sense that it has zero Lebesgue measure. However, a classical result of Jarník [20] states that this set is in some sense as large as it can be in that it is of full Hausdorff dimension, i.e. $\dim \mathbf{Bad} = \dim \mathbb{R} = 1$. In later works of Davenport [15], Schmidt [28], Pollington & Velani [26], and others, this result was generalized to badly approximable points in $\mathbb{R}^N$, $N > 1$. In particular, the result in [26] states that the set $\mathbf{Bad}(i_1, \ldots, i_N)$ of points $(x_1, \ldots, x_N) \in \mathbb{R}^N$ for which

$$\exists\, c > 0 \ \text{ s.t. } \ \max\{||qx_1||^{\frac{1}{i_1}}, \ldots ||qx_N||^{\frac{1}{i_N}}\} \geqslant \frac{c}{q} \ \forall q \in \mathbb{N} \tag{1}$$

has full Hausdorff dimension N, where $i_1, \ldots, i_N$ are any strictly positive real numbers satisfying $i_1 + \ldots + i_N = 1$. Here, $||\,.\,||$ denotes the distance to the nearest integer. Finally, in [22] a quite general theory of badly approximable sets was constructed. It allows one to establish full Hausdorff dimension results for a quite general class of sets living in arbitrary compact metric spaces as long as certain structural conditions are satisfied. The framework developed encompasses the results of both Jarník and Pollington & Velani as described above. Broadly speaking, the sets considered in [22] consist of points in a metric space that avoid a given family of subsets of the metric space; that is, points which cannot be easily approximated by this family of subsets. Naturally, such sets were also referred to as 'badly approximable'. We give further details of this theory in Section 6.

*Research supported by EPSRC Grant EP/L005204/1

Recently, various sets were introduced within the theory of Diophantine approximation that on one hand could be naturally associated with the notion of badly approximable sets, but on the other hand do not seem to be covered by the framework of [22]. For example, in the landmark paper [6] the authors showed that the set $\mathbf{Bad}(i,j)$ intersected with any vertical line in $\mathbb{R}^2$ is either empty or has Hausdorff dimension equal to one. Whether this intersection is empty or has full dimension depends only upon a Diophantine property of the vertical line parameter. Later, Beresnevich [9] showed that for any non-degenerate manifold $\mathcal{M} \subset \mathbb{R}^N$,

$$\dim(\mathbf{Bad}(i_1,\ldots,i_N) \cap \mathcal{M}) = \dim \mathcal{M},$$

or in other words $\mathbf{Bad}(i_1,\ldots,i_N) \cap \mathcal{M}$ has full Hausdorff dimension.

One of the aims of this paper is to develop a framework in the theory of badly approximable sets which will cover these new results. In addition, we will show in detail (see Section 7) exactly how our theory can be used to attack various problems in the field of Diophantine approximation, some of them old and some of them previously open.

Our methodology appeals to the idea of *generalised Cantor sets in* $\mathbb{R}^N$, which first appeared within the proofs of [6] and whose concept was developed in the subsequent paper [7]. The construction of generalised Cantor sets has formed a basis for resolving various difficult problems in the field of Diophantine approximation. Many of these problems had proven resistant to previous methods. For example, in [3] generalised Cantor sets were utilised to show that the set of points $(x,y) \in \mathbb{R}^2$ satisfying

$$\liminf_{q\to\infty} q \cdot \log q \cdot \log\log q \cdot ||qx|| \cdot ||qy|| > 0 \tag{2}$$

is of maximal Hausdorff dimension 2 - a set not falling within the scope of [22]. This result represented significant progress in the investigation towards the famous Littlewood Conjecture, which postulates that the set of $(x,y) \in \mathbb{R}^2$ satisfying (2), but with the '$\log q \cdot \log\log q$' term removed, is empty.

Whilst the Littlewood Conjecture is considered one of the most profound and evasive problems in all of Diophantine approximation, in recent years there has been much interest in a relatively new and related problem. In 2004, de Mathan and Teulié [24] proposed the following. Let $\mathcal{D} = (d_n)_{n\in\mathbb{N}}$ be a sequence of positive integers greater or equal to 2 and let

$$D_0 := 1; \quad D_n := \prod_{k=1}^{n} d_k.$$

Then, define the 'pseudo-norm' function $|\cdot|_{\mathcal{D}} : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ by

$$|q|_{\mathcal{D}} = \min\{D_n^{-1} : q \in D_n\mathbb{Z}\}.$$

If $\mathcal{D} = (p)_{n\in\mathbb{N}}$ is a constant sequence for some prime number p then $|\cdot|_{\mathcal{D}} = |\cdot|_p$ is the usual $p-$adic norm. The de Mathan-Teulié Conjecture, often referred to as the 'Mixed' Littlewood Conjecture, is the assertion that for any sequence $\mathcal{D}$ and for every $x \in \mathbb{R}$ we have

$$\liminf_{q\to\infty} q \cdot |q|_{\mathcal{D}} \cdot ||qx|| \ = \ 0. \tag{3}$$

In [7] generalised Cantor sets were utilised to show that the set of real numbers $x \in \mathbb{R}$ satisfying

$$\liminf_{q\to\infty} q \cdot \log q \cdot \log\log q \cdot |q|_{\mathcal{D}} \cdot ||qx|| > 0 \tag{4}$$

is of maximal Hausdorff dimension. For arbitrary sequences $\mathcal{D}$, this result represents the state of the art. However, an application of the framework developed in our paper shows that for sequences $\mathcal{D}$ growing sufficiently quickly statement (4) can be significantly improved. In particular, we show that for any monotonic function $g : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ tending to infinity and every sequence $\mathcal{D} = (d_i)_{i\in\mathbb{N}}$ such that

$$\lim_{i\to\infty} \frac{g(D_{i+1})}{g(D_i)} = \infty,$$

the set of real numbers $x \in \mathbb{R}$ satisfying

$$\liminf_{q\to\infty} q \cdot g(q) \cdot |q|_{\mathcal{D}} \cdot ||qx|| > 0 \tag{5}$$

is of maximal Hausdorff dimension. Note that we may choose $\mathcal{D}$ and g so that g tends to infinity as slowly as we wish. The only previously known result of this type 'beating' the rate of approximation in (4) was proved in [7], where it was shown that the set of $x \in \mathbb{R}$ satisfying (5) with $g(q) = \log\log q \cdot \log\log\log q$ has maximal Hausdorff dimension in the specific case $\mathcal{D} = \{2^{2^n}\}_{n=0}^{\infty}$.

Our paper also extends the concept of generalised Cantor sets to the setting of general metric spaces. This allows us to utilise modern techniques in setups to which they did not previously extend. As one of the applications considered in this paper we consider the space $\mathbb{Z}_p$ of p-adic integers. For $N \in \mathbb{N}$, the set $\mathbf{Bad}_p(N)$ of so-called *badly approximable p-adic vectors* is defined as the collection of points $(x_1, \ldots, x_N) \in \mathbb{Z}_p^N$ for which there exists a constant $c > 0$ satisfying

$$\max\{|qx_1 - r_1|_p, \ldots |qx_N - r_N|_p\} \geqslant c \cdot \max\{|r_1|, \ldots, |r_N|, |q|\}^{-\frac{N+1}{N}} \tag{6}$$

for every $(r_1, \cdots, r_N, q) \in \mathbb{Z}^N \times \mathbb{Z} \setminus \{0\}$. The set $\mathbf{Bad}_p(1)$ was shown to have maximal Hausdorff dimension by Abercrombie [2] in 1995. In 2006, this result was extended using the broad framework of [22], where it was shown that the set $\mathbf{Bad}_p(N)$ has maximal Hausdorff dimension N. However, nothing is known about any stronger properties of $\mathbf{Bad}_p(N)$, such as whether it is winning with respect to Schmidt's game, which we now introduce. We show that at the very least $\mathbf{Bad}_p(N)$ satisfies the amenable properties enjoyed by the 'winning sets' occurring in Schmidt's game. Establishing these properties had previously appeared out of reach.

We also find answers to some other new problems from the field of Diophantine approximation, such as questions relating to the $\times 2, \times 3$ problem, and questions relating to the behaviour of the Lagrange constant of multiples of a given real number as posed in [13].

1.2 Winning sets and countable intersections

Given a ball B we write $\mathrm{rad}(B)$ and $\mathrm{diam}(B)$ for the radius and the diameter of B respectively. By $\mathrm{cent}(B)$ we denote the center of B.

Another remarkable property of the set **Bad** was discovered by Schmidt in a series of works finalised in [27]. It can be described in terms of Schmidt's so called (α, β)-*game*. Suppose two players Alice (A) and Bob (B) play the following game with two fixed real parameters $0 < \alpha, \beta < 1$. Bob starts by choosing an arbitrary closed ball $B_1 \subset \mathbb{R}^N$. Then Alice and Bob take turns in choosing closed balls in a nested sequence (Alice chooses balls A_i and Bob chooses balls B_i),

$$B_1 \supset A_1 \supset B_2 \supset A_2 \supset \cdots,$$

whose radii satisfy

$$\mathrm{rad}(A_i) = \alpha \cdot \mathrm{rad}(B_i), \quad \mathrm{rad}(B_{i+1}) = \beta \cdot \mathrm{rad}(A_i), \quad \forall i \in \mathbb{N}.$$

A set $E \subset \mathbb{R}^N$ is called (α, β)-*winning* if Alice has a strategy that ensures that

$$\bigcap_{i=1}^{\infty} B_i \in E.$$

Finally, we say that $E \subset \mathbb{R}^N$ is α-*winning* if it is (α, β)-winning for all $0 < \beta < 1$ and *winning* if it is α-winning for some $\alpha \in (0, 1)$.

Surprisingly, Schmidt was able to show that the set **Bad** is winning as a subset of $\mathbb{R}$, and further, that all winning sets in Euclidean space satisfy some remarkable properties:

(W1) Any winning set is dense and has full Hausdorff dimension.

(W2) Any countable intersection of α-winning sets is α-winning.

(W3) The image of any winning set under a bi-Lipschitz map is again winning.

Many other sets, including $\mathbf{Bad}(1/N, 1/N, \ldots, 1/N)$ (which for brevity we refer as $\mathbf{Bad}_N$), have been proven to be winning [29]. Most recently, in an exceptional paper by An [1] it was shown that the set $\mathbf{Bad}(i, j)$ is winning. This provided a second proof of the Badziahin-Pollington-Velani Theorem, the main result of [6], which in turn established a long standing conjecture of Schmidt. It appears that many sets that according to [22] fall into the class of badly approximable sets are indeed winning.

Several variations of Schmidt's (α, β)-game have been suggested whose analogous winning sets still satisfy the properties (W1) – (W3) of classical winning sets. On the further development of the subject we refer the reader to [11, 21, 23, 29] and the references therein, and to Section 5 of this paper for a partial overview. One disadvantage of working with topological games of this type is that it is often quite difficult to prove a set is winning. A major aim of this paper is to develop a variation of the class of winning sets such that properties (W1) – (W3) are still satisfied for sets in this new class and the conditions for inclusion this class are rather easier to check. In particular, we define so called *Cantor-winning* sets in Section 4; a classy of sets each of whom contain a generous supply of generalised Cantor sets.

As examples, the set consisting of the points in $\mathbf{Bad}(i, j)$ lying on certain vertical lines which appeared in [6] turns out to be Cantor-winning, and so does the set of points in $\mathbf{Bad}(i_1, \ldots, i_N)$ lying on non-degenerate curves as described in [9]. Therefore, these sets fall into a class of sets satisfying properties (W1) – (W3). We discuss these results in more details in Section 7, along with some other far reaching applications.

1.3 The idea of generalised Cantor sets

The basic premise for the construction of generalised Cantor sets in an arbitrary metric space is the standard middle-third Cantor set construction. We now describe this process and then discuss what requirements should be satisfied in order to generalize the construction to an arbitrary metric space. The classical Cantor set is realised as follows. We start with the unit interval $I_0 = [0, 1]$. The first step of the process is to split the interval I_0 into three intervals of equal length and remove the open middle interval. This leaves a union $I_1 = [0, 1/3] \cup [1/3, 1]$ of two disjoint closed intervals which survive the first step. We recursively repeat this procedure for each of the remaining intervals, each time removing the open middle third from every interval in the union, to produce a sequence $(I_2, I_3, \ldots)$ of sets. Each I_i will consist precisely of the disjoint union of the 2^i closed intervals that survive the i-th step of the removal procedure. The classical middle-third Cantor set $\mathcal{K}$ is then defined as

$$\mathcal{K} := \bigcap_{i=0}^{\infty} I_i.$$

The set $\mathcal{K}$ is well known to be uncountable and have Hausdorff dimension $\log 2/\log 3$. Surely for I_0 we can take any interval instead of $[0, 1]$ and the final set $\mathcal{K}$ will still satisfy the same properties.

In an arbitrary metric space X the (metric) balls will play the role of intervals in $\mathbb{R}$. One needs to define the rules describing how each surviving ball should be split into smaller pieces in the next step of the construction. For example, when $X = \mathbb{R}$ we may generalise the set $\mathcal{K}$ by splitting intervals into R closed pieces of equal length at each step for some $R \geq 3$, or even varying the number of intervals created during each step of the procedure. In $\mathbb{R}^N$ we can take square boxes (that is, balls in the sup-norm metrics) and split them into R^N smaller boxes. For a general metric space X we will need to describe how every ball is split into smaller balls. In order for such a process to be meaningful (or even possible) we must enforce some kind of structure on X which allows for such a splitting procedure to take place. In Section 2 we define an extremely general class of metric spaces possessing such a *splitting structure*.

Returning to $\mathcal{K}$ for a moment, we may also generalise its construction by varying the number of intervals removed at each step. However, this should be done with care. For example, if in a classical Cantor set construction instead of one interval we remove two of them on each step (let's say a middle and left one) then every step will leave just one interval: $I_1 = [2/3, 1]$, $I_2 = [8/9, 1]$ and so on. In this case $\bigcap_{i=0}^{\infty} I_i$ is a single point, which is probably not as interesting as $\mathcal{K}$. For this reason we need to control the number of intervals produced in the splitting process together with the number of removed intervals in each step in order to get a non-trivial generalised Cantor set at the end. We provide reasonable restrictions on these numbers in Section 3, although they are essentially the same as in [7]. The key point of the described procedure is that we shall allow the number of intervals removed at each step of the removal process to depend on the entirety of the construction thus far, not just upon the specific step as in the classical Cantor set construction.

2 Splitting structure and metric spaces

We now describe sufficient conditions on a metric space X for a generalised Cantor construction to be possible. Denote by $\mathcal{B}(X)$ the set of all closed (metric) balls in X and by $\mathcal{B}^*(X)$ the set of all finite subsets of $\mathcal{B}(X)$.

We define a *splitting structure* on a metric space X (with metric $\mathbf{d}$) as a quadruple $(X, \mathcal{S}, U, f)$, where

- $U \subset \mathbb{N}$ is an infinite multiplicatively closed set;
- $f : U \to \mathbb{N}$ is a completely multiplicative arithmetic function;
- $\mathcal{S} : \mathcal{B}(X) \times U \to \mathcal{B}^*(X)$ is a map defined in such a way that for every ball $B \in \mathcal{B}(X)$ and $u \in U$, the set $\mathcal{S}(B, u)$ consists solely of balls $b_i \subset B$ of radius $\text{rad}(B)/u$.

Additionally, we require all these objects to be linked by the following properties

(S1) $\#\mathcal{S}(B, u) = f(u)$;

(S2) If $b_1, b_2 \in \mathcal{S}(B, u)$ and $b_1 \neq b_2$ then b_1 and b_2 may only intersect on the boundary; i.e.

$$\mathbf{d}(\text{cent}(b_1), \text{cent}(b_2)) \geqslant \frac{2 \cdot \text{rad}(B)}{u};$$

(S3) For all $u, v \in U$,

$$\mathcal{S}(B, uv) = \bigcup_{b \in \mathcal{S}(B,u)} \mathcal{S}(b, v).$$

Remark 2.1. Not all metric spaces possess an interesting splitting structure. For example, it is easy to check that if $f \not\equiv 1$ then X must be infinite. On the other hand in the case $f \equiv 1$ we always have that $\mathcal{S}(B,u)$ consists of one ball. This case is not very interesting and we call such a splitting structure *trivial.* Furthermore, given a metric space X, there usually exist some restrictions on the growth of f for the splitting structure $(X,\mathcal{S},U,f)$ to exist. For example, when $X = \mathbb{R}^N$ properties (S1) and (S2) imply that we must have $f(u) \leqslant u^N$.

Note also that $\mathcal{S}(B,u)$ does not necessarily form a cover of B. However, in the cases we are most interested in this property will be satisfied.

2.1 Some examples

(a) Let $X = \mathbb{R}^N$ with $\mathbf{d}(\mathbf{x},\mathbf{y}) := |\mathbf{x}-\mathbf{y}|_\infty$, $U = \mathbb{N}$, $f(u) = u^N$ and $\mathcal{S}(B,u)$ be defined as follows: B is cut into u^N equal square boxes which edges have length u times less than the edges length of B. One can easily check that $(\mathbb{R}^N,\mathcal{S},\mathbb{N},f)$ satisfies properties (S1) – (S3). We call this the *canonical* splitting structure for $\mathbb{R}^N$.

(b) Let $X = \mathbb{Q}_p^N$ with $\mathbf{d}(\mathbf{x},\mathbf{y}) := \max_{1\leqslant i\leqslant N}\{|x_i - y_i|_p\}$, $U = \{p^k \,:\, k \in \mathbb{Z}_{\geqslant 0}\}$, $f(p^k) = p^{Nk}$ and $\mathcal{S}(B,p^k)$ be defined as the set of all disjoint balls in B of radius $\mathrm{rad}(B)/p^k$. Again properties (S1) – (S3) are easily verified, so $(\mathbb{Q}_p^N,\mathcal{S},U,f)$ is a splitting structure. We call this splitting structure *canonical* for $\mathbb{Q}_p^N$.

(c) We give one more exotic example. Let $X = \mathbb{R}$, $U = \{3^k \,:\, k \in \mathbb{Z}_{\geqslant 0}\}$, $f(3^k) = 2^k$. Define $\mathcal{S}(B,3)$ as follows: we divide the interval B into 3 pieces of equal length and remove the open third in the middle. $\mathcal{S}(B,3^k)$ for $k > 1$ is constructed inductively with help of property (S3). It is easily verified that $(\mathbb{R},\mathcal{S},U,f)$ indeed forms a splitting structure.

We will refer to these examples throughout the paper.

2.2 The set $A_\infty(B)$

A splitting structure on a metric space naturally exhibits a Cantor-like structure. For $u \in U$ and $B \in \mathcal{B}(X)$ define the set

$$A_u(B) := \bigcup_{b\in\mathcal{S}(B,u)} b.$$

By property (S3), if $u,v \in U$ then $A_{uv}(B) \subset A_u(B)$. Moreover, if X is complete then for every sequence $(u_i)_{i\in\mathbb{N}}$ with $u_i \in U$ such that $u_i \mid u_{i+1}$ and $u_{i+1}/u_i \in U$ the set $\bigcap_{i=1}^\infty A_{u_i}(B)$ is non-empty. Further, it has non-empty intersection with each ball from $\mathcal{S}(B,u_i)$ and the following property also holds.

Theorem 2.1. *Let $(X,\mathcal{S},U,f)$ be a splitting structure. Then for every two sequences $(u_i)_{i\in\mathbb{N}}$ and $(v_i)_{i\in\mathbb{N}}$ with $u_i,v_i \in U$ such that $u_i \mid u_{i+1}, v_i \mid v_{i+1}$ and $u_i, v_i \xrightarrow{i\to\infty} \infty$ one has*

$$\bigcap_{i=1}^\infty A_{u_i}(B) = \bigcap_{i=1}^\infty A_{v_i}(B).$$

Proof. Suppose the contrary is true, and without loss of generality assume that $\bigcap_{i=1}^\infty A_{u_i}(B)$ is non-empty; that is, suppose there exists $x \in X$ and $k \in \mathbb{N}$ such that

$$x \in \bigcap_{i=1}^\infty A_{u_i}(B), \quad \text{but} \quad x \notin A_{v_k}(B).$$

Since $A_{v_k}(B)$ is a finite union of closed sets and is therefore closed, there exists a real number $\delta > 0$ and a ball $B(x,\delta) \in \mathcal{B}(X)$ around x such that

$$B(x,\delta) \cap A_{v_k}(B) = \emptyset.$$

But, by the construction of the sets $A_{u_i}(B)$ there exists $m \in \mathbb{N}$ and a ball $b \in \mathcal{S}(B, u_m)$ of radius less than $\delta/2$ such that $x \in b$. Therefore, $b \subset B(x,\delta)$. Finally, by property (S3), the set $A_{u_m v_k}(B)$ has non-empty intersection with b and is itself a subset of $A_{v_k}(B)$. Thus

$$\emptyset \neq b \cap A_{v_k}(B) \subset B(x,\delta) \cap A_{v_k}(B),$$

and we reach a contradiction. □

The crux of Theorem 2.1 is that an infinite intersection $\bigcap_{i=1}^{\infty} A_{u_i}(B)$ depends only on the ball B and the splitting structure on X, but not on the particular sequence (u_i). In further discussion it will always be assumed that X is a complete metric space. This is to ensure that $A_\infty(B)$ has a rich structure, even though $A_\infty(B)$ is now well defined over all metric spaces, not necessarily complete.

It is readily observed that for any trivial splitting structure $A_\infty(B)$ consists of just a single point. Also, one can easily check that for the canonical splitting structure on $\mathbb{R}^N$ and for the canonical splitting structure on $\mathbb{Q}_p^N$ (examples (a) and (b)), we have that $A_\infty(B) = B$. In the case of the more exotic splitting structure from example (c) one can check that $A_\infty(B)$ is a standard middle-third Cantor set $\mathcal{K}(B)$ whose construction starts with the interval $I_0 = B$. As previously discussed, the set $\mathcal{K}(B)$ is compact. Indeed, the set $A_\infty(B)$ is compact for each of the examples (a) - (c). We now demonstrate that this property is actually ubiquitous.

Theorem 2.2. *Let $(X, \mathcal{S}, U, f)$ be a splitting structure. Then, for any ball $B \in \mathcal{B}(X)$ the set $A_\infty(B)$ is compact.*

Proof. For a trivial splitting structure the result is obvious. Therefore, assume that the splitting structure is non-trivial. Fix a parameter $v \in U$ with $v > 1$. Consider a cover $\bigcup_\alpha O_\alpha$ of $A_\infty(B)$ by open sets O_α. To each O_α we may associate a subset of balls from $\bigcup_{i=1}^{\infty} \mathcal{S}(B, v^i)$ such that every ball b from this subset lies entirely inside O_α. In particular, let

$$\mathcal{D}(O_\alpha) := \left\{ b \in \bigcup_{i\in\mathbb{N}} \mathcal{S}(B, v^i) \;:\; b \subset O_\alpha \right\}.$$

Obviously, if $b \in \mathcal{D}(O_\alpha)$ then every ball $b' \in \mathcal{S}(b, v^j)$ for $j \in \mathbb{N}$ is also in $\mathcal{D}(O_\alpha)$.

If for some $i \in \mathbb{N}$ every ball from $\mathcal{S}(B, v^i)$ is in one of the sets $\mathcal{D}(O_\alpha)$ (for some α) then there is a finite subcover of $A_\infty(B)$. Indeed, for every ball b from the finite set $\mathcal{S}(B, v^i)$ we can associate one element O_α from the cover such that $b \in \mathcal{D}(O_\alpha)$. Assume now that this is not the case. Then there exists a sequence $(b_i)_{i\in\mathbb{N}}$ of balls such that $b_i \in \mathcal{S}(B, v^i)$, $b_i \supset b_{i+1}$ such that none of these balls are in $\mathcal{D}(O_\alpha)$ for any α. Since X is complete we have $\bigcap_{i=1}^{\infty} b_i = x$ is a single point. It must be covered by one of the open sets O_α and so there must exist $\epsilon > 0$ such that $B(x,\epsilon) \subset O_\alpha$. Moreover, for i large enough we must have $b_i \subset B(x,\epsilon) \subset O_\alpha$, a contradiction. □

As in the case of the middle-third Cantor set it is desirable to determine the Hausdorff dimension of the set $A_\infty(B)$. In order to compute this in general we require the metric space X to satisfy one further condition.

(S4) There exists an absolute constant $C(X)$ such that any ball $B \in \mathcal{B}(X)$ cannot intersect more than $C(X)$ disjoint open balls of the same radius as B.

It is easy to check that the metric spaces $\mathbb{R}^N$ and $\mathbb{Q}_p^N$ satisfy Condition (S4), and so all of the examples (a) - (c) satisfy it. This condition is sufficient to precisely calculate the Hausdorff dimension of $A_\infty(B)$.

Theorem 2.3. *Let $(X, \mathcal{S}, U, f)$ be a splitting structure. Then for any $B \in \mathcal{B}(X)$ we have*

$$\dim A_\infty(B) \leqslant \liminf_{u\to\infty; u\in U} \frac{\log f(u)}{\log u}.$$

Moreover, if X satisfies condition (S4) then $\log f(u)/\log u$ must be a constant and

$$\dim A_\infty(B) = \frac{\log f(u)}{\log u}.$$

Proof. Determining the upper bound for the Hausdorff dimension is relatively easy. We may simply consider the trivial cover $\mathcal{S}(B,u)$ of $A_\infty(B)$. We have

$$\sum_{b\in\mathcal{S}(B,u)} (\mathrm{rad}(b))^d \asymp f(u)\cdot u^{-d},$$

and so for $d > \liminf \log f(u)/\log u$ one can find a sequence of integers $u_i \in U$ and $\epsilon > 0$ such that

$$\frac{\log f(u_i)}{\log u_i} < d - \epsilon.$$

Therefore,

$$f(u_i)\cdot u_i^{-d} < f(u_i)\cdot u_i^{-\frac{\log f(u_i)}{\log u_i}-\epsilon} = u_i^{-\epsilon} \stackrel{i\to\infty}{\longrightarrow} 0$$

and this gives us a required upper bound on $\dim A_\infty(B)$.

The reverse inequality requires a bit more effort. Consider $u \in U$ and let $d = \log f(u)/\log u$. If we prove that

$$\inf\left\{\sum_i (\mathrm{rad}(B_i))^d \;:\; \bigcup_i B_i \text{ is a cover of } B\right\} > 0 \tag{7}$$

then we will have that $\dim A_\infty(B) \geqslant d$ as required.

Let $\bigcup_\alpha B_\alpha$ be an arbitrary cover of $A_\infty(B)$ by open balls B_α. Then, since $A_\infty(B)$ is compact one can choose a finite subcover $\bigcup_{i=1}^n B_i$. This procedure only decreases the value on the left hand side of (7) and so if we can show that

$$\inf\left\{\sum_i (\mathrm{rad}(B_i))^d \;:\; \bigcup_i B_i \text{ is a finite cover of } B\right\} > 0 \tag{8}$$

then we are done. Given a finite subcover $\bigcup_{i=1}^n B_i$ one may without loss of generality assume that $\mathrm{rad}(B_i) \leqslant \mathrm{rad}(B)$ for each $1 \leqslant i \leqslant n$, for otherwise (8) has an obvious positive infimum equal to $(\mathrm{rad}(B))^d$.

Consider an individual element B_i of the subcover. Let $m_i \in \mathbb{Z}_{\geqslant 0}$ take a value such that

$$\frac{\mathrm{rad}(B)}{u^{m_i+1}} < \mathrm{rad}(B_i) \leqslant \frac{\mathrm{rad}(B)}{u^{m_i}}.$$

Then, take all balls $B_{i,1}, B_{i,2}, \ldots, B_{i,s_i}$ from the collection $\mathcal{S}(B, u^{m_i})$ which have non-empty intersection with B_i. By condition (S4) we must have $s_i \leqslant C(X)$ and so

$$(\mathrm{rad}(B_i))^d \geqslant \frac{1}{C(X)\cdot u^d}\sum_{s=1}^{s_i} (\mathrm{rad}(B_{i,s_i}))^d.$$

Replacing each B_i by $B_{i,1}, \ldots, B_{i,s_i}$ one can easily check that

$$\bigcup_{i=1}^{n} \bigcup_{s=1}^{s_i} B_{i,s} \tag{9}$$

is still a cover of $A_\infty(B)$ and it solely consists of balls from $\bigcup_{i=1}^{\infty} \mathcal{S}(B, u^i)$. The value in (8) does not decrease more than $C(X) \cdot u^d$ times compared to the initial cover $\bigcup_{i=1}^{n} B_i$.

Now notice that by the definition of d, for every ball B' one has

$$(\mathrm{rad}(B'))^d = \sum_{b \in \mathcal{S}(B', u)} (\mathrm{rad}(b))^d.$$

In other words, if in a cover one replaces one ball B' by all balls in $\mathcal{S}(B', u)$, the value (7) does not change. We use this observation and replace if necessary every ball $B_{i,s}$ from (9) by balls from $\mathcal{S}(B_{i,s}, u^{k_{i,s}})$ for some $k_{i,s}$ to guarantee that the set of balls $\mathcal{S}^*$ in the resulting cover are from $\mathcal{S}(B, u^k)$ for some fixed $k \in \mathbb{N}$.

We claim that the set $\mathcal{S}^*$ contains at least $(C(X)+1)^{-1} \# \mathcal{S}(B, u^k)$ balls. Indeed, conditions (S2) and (S4) imply that each ball $b \in \mathcal{S}^*$ intersects no more than $S(X)$ other balls from $\mathcal{S}(B, u^k)$. Therefore if $\#\mathcal{S}^* < (C(X)+1)^{-1} \# \mathcal{S}(B, u^k)$ one can find at least one ball $b' \in \mathcal{S}(B, u^k)$ such that

$$b' \cap \bigcup_{b \in \mathcal{S}^*} b = \emptyset.$$

On the other hand, $b \cap A_\infty(B)$ is nonempty. This contradicts to the fact that $\cup_{b \in \mathcal{S}^*} b$ is a cover of $A_\infty(B)$.

Finally,

$$\sum_{i=1}^{n} (\mathrm{rad}(B_i))^d \geqslant \frac{1}{C(X) \cdot u^d} \cdot \sum_{b \in \mathcal{S}^*} (\mathrm{rad}(b))^d \geqslant \frac{(\mathrm{rad}(B))^d}{C(X)(C(X)+1) \cdot u^d} > 0.$$

The claim is achieved, therefore we get

$$\dim A_\infty(B) \leqslant \frac{\log f(u)}{\log u}.$$

To finish the proof we take an arbitrary $u \in U$ and combine the last statement with the lower bound for $\dim A_\infty(B)$ we got before. □

Corollary 2.4. *If $(X, \mathcal{S}, U, f)$ is a non-trivial splitting structure for which X satisfies condition (S4) and U contains at least two multiplicatively independent numbers, then $f(u)$ must be of the form: $f(u) = u^d$ where $d \in \mathbb{Q}$, $d > 0$.*

Proof. By Theorem 2.3,

$$d = \frac{\log f(u)}{\log u}$$

is a constant. Therefore $f(u) = u^d$. If u_1, u_2 are two multiplicatively independent elements of U, then u_1^d and u_2^d can both be integers only if $d \in \mathbb{Q}$. Finally $d > 0$ since otherwise $u^d \notin \mathbb{Z}$ or $f(u) = 1$. The last condition can only happen for trivial splitting structure $(X, \mathcal{S}, U, f)$. □

Notice that if U is generated by one positive integer number u_0 then for every $u = u_0^n$ we have $f(u) = f(u_0)^n = u^d$ where $d = \frac{\log f(u_0)}{\log u_0}$. Again f is of the form $f(u) = u^d$, however in this case we do not necessarily have that $d \in \mathbb{Q}$.

3 Generalised Cantor sets

Let $(X, \mathcal{S}, U, f)$ be a splitting structure on X. We now introduce the precise definition of generalised Cantor sets in the context of this splitting structure, for which we appeal heavily to the ideas presented in [7].

Fix some closed ball $B \in \mathcal{B}(X)$, let

$$\mathbf{R} := (R_n)_{n \in \mathbb{Z}_{\geqslant 0}}, \ R_n \in U$$

be a sequence of natural numbers and let

$$\mathbf{r} := (r_{m,n}), \ m, n \in \mathbb{Z}_{\geqslant 0}, \ m \leqslant n$$

be a two parameter sequence of real numbers.

Construction. We start by considering the set $\mathcal{S}(B, R_0)$. The first step in the construction of a generalised Cantor set involves the removal of at most $r_{0,0}$ balls b from $\mathcal{S}(B, R_0)$. We call the resulting set $\mathcal{B}_1$. Balls in $\mathcal{B}_1$ will be referred as (level one) survivors. Note that we do not specify the removed balls, we just give an upper bound for their number. For consistency we also define $\mathcal{B}_0 := \{B\}$.

In general, for $n \geqslant 0$, given a collection $\mathcal{B}_n$ we construct a nested collection $\mathcal{B}_{n+1}$ using the following two operations:

- Splitting procedure: Compute the collection of candidate balls

$$\mathcal{I}_{n+1} := \bigcup_{B_n \in \mathcal{B}_n} \mathcal{S}(B_n, R_n).$$

- Removing procedure: For each ball $B_n \in \mathcal{B}_n$ we remove at most $r_{n,n}$ balls $B_{n+1} \in \mathcal{S}(B_n, R_n)$ from $\mathcal{I}_{n+1}$. Let $\mathcal{I}_{n+1}^n \subseteq \mathcal{I}_{n+1}$ be the collection of balls that remain. Next, for each ball $B_{n-1} \in \mathcal{B}_{n-1}$ we remove at most $r_{n-1,n}$ balls $B_{n+1} \in \mathcal{S}(B_{n-1}, R_n R_{n-1}) \cap \mathcal{I}_{n+1}^n$. Let $\mathcal{I}_{n+1}^{n-1}$ be the collection of balls that remain. In general for each $B_{n-k} \in \mathcal{B}_{n-k}$ $(1 \leqslant k \leqslant n)$ we remove at most $r_{n-k,n}$ balls $B_{n+1} \in \mathcal{S}(B_{n-k}, \prod_{i=0}^{k} R_{n-i}) \cap \mathcal{I}_{n+1}^{n-k+1}$ and define $\mathcal{I}_{n+1}^{n-k} \subseteq \mathcal{I}_{n+1}^{n-k+1}$ to be the collection of balls that remain. Finally, $\mathcal{B}_{n+1} := \mathcal{I}_{n+1}^0$ then becomes the desired collection of (level $n+1$) survivors.

The two operations above allow us to construct a nested sequence of collections $\mathcal{B}_n$ of closed balls. Consider the limit set

$$\mathcal{K}(B, \mathbf{R}, \mathbf{r}) := \bigcap_{i=1}^{\infty} \bigcup_{b \in \mathcal{B}_n} b.$$

The set $\mathcal{K}(B, \mathbf{R}, \mathbf{r})$ will be referred to as a *generalised* $(B, \mathbf{R}, \mathbf{r})$-*Cantor set* on X.

Note that the triple $(B, \mathbf{R}, \mathbf{r})$ does not uniquely determine $\mathcal{K}(B, \mathbf{R}, \mathbf{r})$. There is a large degree of freedom in the choice of balls B_{n+1} removed in the construction procedure. Consequently, one can look at the property of being a generalised $(B, \mathbf{R}, \mathbf{r})$-Cantor set as a property of a given set $\mathcal{K} \subset X$, rather than as a self contained definition: we say a set $\mathcal{K}$ is *a generalised Cantor set* if it can be constructed by the procedure described above for some triple $(B, \mathbf{R}, \mathbf{r})$. In this case, we may refer to $\mathcal{K}$ as being $(B, \mathbf{R}, \mathbf{r}) -$ *Cantor* if we wish to make such a triple explicit and write $\mathcal{K} = \mathcal{K}(B, \mathbf{R}, \mathbf{r})$.

3.1 Properties of $\mathcal{K}(B, \mathbf{R}, \mathbf{r})$

Generalized $(B, \mathbf{R}, \mathbf{r})$-Cantor sets in any complete metric space X satisfy the same desirable properties as proved in [7]. Furthermore, many of the proofs translate from the Euclidean setting to the case of arbitrary metric spaces with only slight modification. We now exhibit these properties, but will provide the proof only if it significantly differs from the analogous methods outlined in [7].

Theorem 3.1 (See Theorem 3 in [7]). *Given a generalised Cantor set $\mathcal{K}(B, \mathbf{R}, \mathbf{r})$ in a complete metric space X, let*

$$t_0 := f(R_0) - r_{0,0} \tag{10}$$

and for $n \geqslant 1$ let

$$t_n := f(R_n) - r_{n,n} - \sum_{k=1}^{n} \frac{r_{n-k,n}}{\prod_{i=1}^{k} t_{n-i}} . \tag{11}$$

Suppose that $t_n > 0$ for all $n \in \mathbb{Z}_{\geqslant 0}$. Then,

$$\mathcal{K}(\mathrm{B}, \mathbf{R}, \mathbf{r}) \neq \emptyset \ .$$

Theorem 3.2 (See Theorem 4 in [7]). *Let a complete metric space X satisfy condition (S4). Given a generalised Cantor set $\mathcal{K}(\mathrm{B}, \mathbf{R}, \mathbf{r}) \subset X$, suppose that the parameters $\mathbf{R}$ and $\mathbf{r}$ satisfy the following conditions:*

- *$f(R_n) \geqslant 4$ for all $n \in \mathbb{Z}_{\geqslant 0}$;*
- *for every $\delta > 0$ there exists $n(\delta)$ such that for every $n > n(\delta)$,*

$$\prod_{i=0}^{n} R_i^{\delta} > R_n; \tag{12}$$

- *For every $n \in \mathbb{Z}_{\geqslant 0}$,*

$$\sum_{k=0}^{n} \left(r_{n-k,n} \prod_{i=1}^{k} \left(\frac{4}{f(R_{n-i})} \right) \right) \leqslant \frac{f(R_n)}{4}. \tag{13}$$

Then

$$\dim \mathcal{K}(\mathrm{B}, \mathbf{R}, \mathbf{r}) \geqslant \liminf_{n \to \infty} (\dim A_\infty(B) - \log_{R_n} 2).$$

Remark 3.1. It appears possible to the authors that condition (12) is not absolutely necessary, but we leave this consideration to the enthusiastic reader. The corresponding Theorem 4 from [7] does not have it. However, its proof contains a mistake[1]. At the very least, (12) is crucial for Lemma 3.1 as it becomes false without it unless an additional hypothesis is assumed. Roughly speaking, the condition ensures that the sequence $\mathbf{R}$ does not grow too rapidly. Whilst the proof of Theorem 3.2 is very similar to Theorem 4 of [7], we consider it to be quite important. For this reason, and for the sake of completeness, we briefly outline its proof here. Finally, note that we use the convention that the product term in (13) is one when '$k = 0$'.

[1] It is on page p.2787 and we thank the referee for pointing that the inequality in question can be salvaged up to a factor of 2 if it is assumed that $t_n = R_n/2$, which is the case of the lemma needed for the proof of Theorem 4 of [7].

Prior the proof we give a definition of *local Cantor sets*, which provide the means by which one can prove most of the results in this section. A generalised Cantor set $\mathcal{K}(\mathrm{B},\mathbf{R},\mathbf{r})$ is said to be *local* if $r_{m,n}=0$ whenever $m\neq n$. Furthermore, we write $\mathcal{LK}(\mathrm{B},\mathbf{R},\mathbf{s})$ for $\mathcal{K}(\mathrm{B},\mathbf{R},\mathbf{r})$ where

$$\mathbf{s}:=(s_n)_{n\in\mathbb{Z}_{\geqslant 0}} \quad \text{and} \quad s_n:=r_{n,n}.$$

We will also need the following version of the mass distribution principle for general metric spaces X, a powerful tool for calculating lower bounds for Hausdorff dimension.

Mass Distribution Principle. *Let μ be a probability measure supported on a subset E of a metric space X. Suppose there are positive constants a, s and l_0 such that*

$$\mu(B)\leqslant a\,\mathrm{diam}(B)^s\,, \tag{14}$$

for any closed set B with $\mathrm{diam}(B)\leqslant l_0$. *Then,* $\dim E\geqslant s$.

One can check that it is sufficient to verify property (14) for all balls $B\in\mathcal{B}(X)$. Indeed, assume that it is satisfied for balls. Consider an arbitrary set $S\subset X$ of diameter $\mathrm{diam}(S)\leqslant l_0/2$. It is covered by a ball B with $\mathrm{rad}(B)\leqslant\mathrm{diam}(S)$, so $\mathrm{diam}(B)\leqslant l_0$. Then we have

$$\mu(S)\leqslant\mu(B)\leqslant a\cdot\mathrm{diam}(B)^s\leqslant a\cdot 2^s\cdot\mathrm{diam}(S)^s.$$

Therefore Property (14) is satisfied then for an arbitrary set S with parameters $a':=a\cdot 2^s$, $s':=s$ and $l_0':=l_0/2$. It follows that $\dim E\geqslant s'=s$.

The final prerequisites for the proof of Theorem 3.2 are a lower bound for the Hausdorff dimension of local Cantor sets and a proof that certain generalised Cantor sets contain sufficiently permeating local Cantor sets.

Lemma 3.1. *Given $\mathcal{LK}(\mathrm{B},\mathbf{R},\mathbf{s})$, suppose that*

$$t_n:=f(R_n)-s_n>0 \quad \forall\ n\in\mathbb{Z}_{\geqslant 0}\,.$$

Furthermore, suppose the values s_n and R_n satisfy the following conditions: there are positive constants s and n_0 such that for all $n>n_0$

$$R_n^s\leqslant t_n \tag{15}$$

and for every $\delta>0$ there exists $n(\delta)>0$ such that inequality (12) *is satisfied. Then*

$$\dim\mathcal{LK}(B,\mathbf{R},\mathbf{s})\geqslant s.$$

Remark 3.2. As we have already mentioned, condition (12) is necessary here. For example, for the canonical splitting structure on $\mathbb{R}$ one can take $R_n=R^{2^n}$, $t_n=[R_n^s]$. If for each interval $b\in\mathcal{B}_n$ all surviving intervals $b_{n+1}\in\mathcal{B}_{n+1}$ such that $b_{n+1}\subset b$ are packed inside one interval of length $t_{n+1}\cdot\mathrm{diam}(b_{n+1})$ the Hausdorff dimension of $\mathcal{K}(B,\mathbf{R},\mathbf{r})$ is strictly less than s. On the other hand if one replaces (15) by a stronger condition $t_n\geqslant f(R_n)/2$ then (12) can be avoided. We leave the proof of these facts to the reader.

Proof. We construct a probability measure μ supported on $\mathcal{LK}(B,\mathbf{R},\mathbf{s})$ in the standard manner. For any $B_n\in\mathcal{B}_n$, we attach a weight $m(B_n)$ defined recursively as follows.
For $n=0$ let

$$m(B_0)\ :=\frac{1}{\#\mathcal{B}_0}=1,$$

and for $n \geqslant 1$ define

$$m(B_n) := \frac{m(B_{n-1})}{\#\{B \in \mathcal{B}_n \ : \ B \subset B_{n-1}\}}, \tag{16}$$

where $B_{n-1} \in \mathcal{B}_{n-1}$ is the unique ball such that $B_n \subset B_{n-1}$. This procedure inductively defines a mass on any ball appearing in the construction of $\mathcal{LK}(B, \mathbf{R}, \mathbf{s})$. In fact, it can be easily demonstrated via induction that for every $B_n \in \mathcal{B}_n$ we have

$$m(B_n) \leqslant \prod_{i=0}^{n-1} t_i^{-1}. \tag{17}$$

Then we can define a measure μ of any Borel subset of X with support on $\mathcal{LK}(B, \mathbf{R}, \mathbf{s})$ in the following way. If E is any Borel subset of X then

$$\mu(E) := \lim_{n\to\infty} \inf \left\{ \sum_i m(B_i) \ : \ E \subset \bigcup_i B_i \text{ and } B_i \in \mathcal{B}_n \right\}.$$

We will call such a measure *a canonical measure* on $\mathcal{LK}(B, \mathbf{R}, \mathbf{s})$. It remains to show that μ satisfies (14). Consider an arbitrary ball E of radius not bigger than $\mathrm{rad}(B)$. Then there exists a positive integer parameter m such that

$$\frac{\mathrm{rad}(B)}{\prod_{i=0}^{m} R_i} < \mathrm{rad}(E) \leqslant \frac{\mathrm{rad}(B)}{\prod_{i=0}^{m-1} R_i}. \tag{18}$$

Now we estimate $\mu(E)$. First, notice that we have

$$\mu(E) \leqslant \sum_{b \in \mathcal{B}_m : \ b \cap E \neq \emptyset} m(b).$$

By Property (S4) there are at most $C(X)$ balls $b \in \mathcal{B}_m$ such that $b \cap E \neq \emptyset$. This, together with (17), gives us the upper bound

$$\mu(E) \leqslant C(X) \cdot \prod_{i=0}^{m-1} t_i^{-1} = C(X) \cdot \prod_{i=0}^{m-1} \frac{R_i^s}{t_i} \Big/ \prod_{i=0}^{m-1} R_i^s \overset{(12)}{\leqslant} C(X) \cdot C_1(\delta) \frac{\mathrm{rad}(B)^{s-s\delta}}{\prod_{i=0}^{m} R_i^{s-s\delta}} \cdot \prod_{i=0}^{m-1} \frac{R_i^s}{t_i},$$

where

$$C_1(\delta) = \mathrm{rad}(B)^{s\delta - s} \cdot \max_{1 \leqslant j \leqslant n(\delta)} \left\{ \left(\frac{R_j}{\prod_{i=0}^{j} R_i^{\delta}} \right)^s, 1 \right\}$$

is a constant independent of the choice of E. We continue with the chain of upper inequalities to get

$$\mu(E) \overset{(18)}{\leqslant} C(X) \cdot C_1(\delta) \cdot C_2 \cdot (\mathrm{rad}(E))^{s-s\delta},$$

where

$$C_2 = \max_{1 \leqslant j \leqslant n_0} \left\{ \prod_{i=0}^{j} \frac{R_i^s}{t_i}, 1 \right\}$$

is again independent on the choice of E. By applying the Mass Distribution Principle we conclude that $\dim \mathcal{LK}(B, \mathbf{R}, \mathbf{s}) \geqslant s - s\delta$. Since δ is arbitrary the lemma is proven. □

Lemma 3.2 (See Proposition 3 in [7]). *Let $\mathcal{K}(\mathrm{I}, \mathbf{R}, \mathbf{r})$ be as in Theorem 3.2. Then there exists a local Cantor set*

$$\mathcal{LK}(\mathrm{I}, \mathbf{R}, \mathbf{s}) \subset \mathcal{K}(\mathrm{I}, \mathbf{R}, \mathbf{r}),$$

where

$$\mathbf{s} := (s_n)_{n \in \mathbb{Z}_{\geqslant 0}} \quad \text{with} \quad s_n := \tfrac{1}{2} f(R_n).$$

Proof of Theorem 3.2.

By Lemma 3.2 we have that

$$\dim \mathcal{K}(B,\mathbf{R},\mathbf{r}) \geqslant \dim \mathcal{LK}(B,\mathbf{R},\mathbf{s}).$$

Now fix some positive $s < \liminf_{n\to\infty}(\dim A_\infty(B) - \log_{R_n} 2)$. Theorem 2.3 gives us that for every n,

$$\dim A_\infty(B) = \frac{\log f(R_n)}{\log R_n} =: d.$$

Then, there exists an integer n_0 such that

$$s \,<\, d - \log_{R_n} 2 \quad \text{for all} \quad n > n_0\,.$$

Also note that

$$t_n = f(R_n) - s_n = \frac{f(R_n)}{2}$$

and

$$R_n^s < \frac{f(R_n)}{2} = t_n \quad \text{for all} \quad n > n_0\,.$$

Therefore, Lemma 3.1 implies that

$$\dim \mathcal{LK}(\mathrm{I},\mathbf{R},\mathbf{s}) \geqslant s\,.$$

The fact that this inequality is true for any $s < \liminf_{n\to\infty}(d - \log_{R_n} 2)$ completes the proof of Theorem 3.2.

Finally we provide the theorem which shows that the intersection of generalised Cantor sets on X is often again a Cantor set.

Theorem 3.3 (See Theorem 5 in [7])**.** *For each integer $1 \leqslant i \leqslant k$, suppose we are given a generalised Cantor set $\mathcal{K}(B,\mathbf{R},\mathbf{r}_i)$. Then*

$$\bigcap_{i=1}^{k} \mathcal{K}(B,\mathbf{R},\mathbf{r}_i)$$

is a $(B,\mathbf{R},\mathbf{r})$-Cantor set, where

$$\mathbf{r} := (r_{m,n}) \quad \textit{with} \quad r_{m,n} := \sum_{i=1}^{k} \lfloor r_{m,n}^{(i)} \rfloor\,.$$

With almost the same proof one can extend this theorem to countable intersections of generalized Cantor sets.

Theorem 3.3*. *For each integer $i \in \mathbb{N}$, suppose we are given a generalised Cantor set $\mathcal{K}(B,\mathbf{R},\mathbf{r}_i)$. Assume that the series*

$$r_{m,n} := \sum_{i=1}^{\infty} \lfloor r_{m,n}^{(i)} \rfloor$$

converges for all pairs $m,n \in \mathbb{N}$ with $m \leqslant n$. Then

$$\bigcap_{i=1}^{\infty} \mathcal{K}(B,\mathbf{R},\mathbf{r}_i)$$

is a $(B,\mathbf{R},\mathbf{r})$-Cantor set with $\mathbf{r} := (r_{m,n})$.

3.2 Images of generalized Cantor sets under bi-Lipschitz maps

Let $\phi : X \to X$ be a bi-Lipschitz homeomorphism; i.e there exists a constant $K > 0$ such that

$$\forall x_1, x_2 \in X, \;\; K^{-1}\mathbf{d}(x_1, x_2) \leqslant \mathbf{d}(\phi(x_1), \phi(x_2)) \leqslant K\mathbf{d}(x_1, x_2).$$

One can easily check that then

$$B(\phi(x), r/K) \subset \phi(B(x, r)) \subset B(\phi(x), Kr).$$

We denote the first (inscribed) ball by $I\phi(B)$ and the second (circumscribed) ball by $E\phi(B)$. We will also need a slightly more restrictive packing condition than property (S4) enforced on the metric space X:

(S5) For each $K \in \mathbb{R}_{>1}$ there exists a constant $C(K, X)$ such that any ball B of radius Kr cannot intersect more than $C(K, X)$ disjoint open balls of radius r.

One can easily check that property (S4) of X follows from property (S5) with $C(X) = C(1, X)$. Finally, note that the spaces X appearing in examples (a) – (c) from Section 2 satisfy condition (S5). We remark that (S5) is usually referred to as the *doubling property* and metric spaces satisfying (S5) are said to be *doubling*.

Theorem 3.4. *Let $(X, \mathcal{S}, U, f)$ be a splitting structure on a complete metric space X satisfying condition (S5). Assume also that $A_\infty(B) = B$ for each ball $B \in \mathcal{B}(X)$. Then for every bi-Lipschitz homeomorphism $\phi : X \to X$ there exists a constant $C > 0$ such that $\phi(\mathcal{K}(B, \mathbf{R}, \mathbf{r}))$ contains some $(I\phi(B), \mathbf{R}, C\mathbf{r})$-Cantor set where*

$$C\mathbf{r} := \{Cr_{m,n} \; : \; m, n \in \mathbb{Z}_{\geqslant 0}, m \leqslant n\}.$$

Remark 3.3. Surely the condition $A_\infty(B) = B$ is quite restrictive. However, it is absolutely essential for the theorem. One can check that the canonical splitting structures for both $\mathbb{R}^n$ and $\mathbb{Q}_p^n$ satisfy that condition. On the other hand the splitting structure $(\mathbb{R}, \mathcal{S}, U, f)$ from example (c) does not satisfy it.

Proof. First, note that since $A_\infty(B) = B$ then for every ball $B \in \mathcal{B}(X)$ and every $R \in U$ we have

$$B = A_\infty(B) \subseteq \bigcup_{b \in \mathcal{S}(B,R)} b \quad \Rightarrow \quad \bigcup_{b \in \mathcal{S}(B,R)} b = B.$$

Since $\mathcal{K}(B, \mathbf{R}, \mathbf{r})$ is a generalized Cantor set we have collections $\mathcal{B}_n, \mathcal{I}_n$ and $\mathcal{I}_n^m$ (for $m < n$) associated with it (see the Cantor set construction algorithm). We now outline the procedure for the construction of the generalised Cantor set inside $I\phi(B)$. Let $\mathcal{B}_0^\phi := \{I\phi(B)\}$. We next inductively construct a nested collection $\mathcal{B}_1^\phi, \mathcal{B}_2^\phi, \ldots, \mathcal{B}_n^\phi, \ldots$. Given a collection $\mathcal{B}_n^\phi$, construct the subsequent collection $\mathcal{B}_{n+1}^\phi$ via the following operations:

- Splitting procedure: Compute the collection

$$\mathcal{I}_{n+1}^\phi := \bigcup_{B_n^\phi \in \mathcal{B}_n^\phi} \mathcal{S}(B_n^\phi, R_n).$$

- Removing procedure: Remove all balls $B_{n+1}^\phi \in \mathcal{I}_{n+1}^\phi$ for which

$$\exists B_{n+1} \in \mathcal{I}_{n+1} \backslash \mathcal{B}_{n+1} \;\; \text{s.t.} \;\; B_{n+1}^\phi \cap \phi(B_{n+1}) \neq \emptyset.$$

By construction we have that

$$\bigcup_{B_n^\phi \in \mathcal{B}_n^\phi} B_n^\phi \subset \phi\left(\bigcup_{B_n \in \mathcal{B}_n} B_n\right)$$

and therefore the set

$$\mathcal{K}^\phi := \bigcap_{i=0}^{\infty} \bigcup_{B^\phi \in \mathcal{B}_i^\phi} B^\phi$$

is a subset of $\phi(\mathcal{K}(B, \mathbf{R}, \mathbf{r}))$.

We will show that the set $\mathcal{K}^\phi$ is indeed $(I\phi(B), \mathbf{R}, C\mathbf{r})$-Cantor for some constant $C > 0$. Consider a ball $B_n^\phi \in \mathcal{B}_n^\phi$. By construction, its radius is $K^{-1} \cdot \prod_{i=0}^{n-1} R_i^{-1} \mathrm{rad}(B)$. If it intersects $\phi(B_n^*)$ for some $B_n^* \in \mathcal{B}_n$ then it also intersects $E\phi(B_n^*)$, whose radius is

$$K \cdot \prod_{i=0}^{n-1} R_i^{-1} \mathrm{rad}(B) = K^2 \cdot \mathrm{rad}(B_n^\phi).$$

Therefore,

$$\mathbf{d}(\mathrm{cent}(B_n^\phi), \mathrm{cent}(E\phi(B_n^*))) \leqslant (1 + K^2)\mathrm{rad}(B_n^\phi).$$

This in turn implies that

$$I\phi(B_n^*) \subset (2 + K^2)B_n^\phi.$$

Since

$$\mathrm{rad}(I\phi(B_n^*)) = \mathrm{rad}(B_n^\phi),$$

it follows from condition (S5) that there are no more than $C(K^2 + 2, X)$ balls $B_n^* \in \mathcal{B}_n$ such that $\phi(B_n^*) \cap B_n^\phi \neq \emptyset$. By the same arguments we deduce that for a fixed ball $B_{n+1} \in \mathcal{B}_{n+1} \backslash \mathcal{I}_{n+1}$ there are at most $C(K^2 + 2, X)$ balls $B_{n+1}^\phi \in \mathcal{I}_{n+1}^\phi$ which have nonempty intersection with $\phi(B_{n+1})$.

Now we construct $\mathcal{I}_{n+1}^{n\phi}$ from $\mathcal{I}_{n+1}^\phi$ by removing all balls $B_{n+1}^\phi \in \mathcal{I}_{n+1}^\phi$ which have nonempty intersection with at least one of the sets $\phi(B_{n+1})$ where $B_{n+1} \in \mathcal{I}_{n+1} \backslash \mathcal{I}_{n+1}^n$. For a fixed ball $B_n \in \mathcal{B}_n$ we have

$$\#\{B_{n+1}^\phi \in \mathcal{I}_{n+1}^\phi : \exists B_{n+1} \in \mathcal{S}(B_n, R_n) \cap (\mathcal{I}_{n+1} \backslash \mathcal{I}_{n+1}^n), \phi(B_{n+1}) \cap B_{n+1}^\phi \neq \emptyset\} \leqslant C(K^2+2, X) r_{n,n}.$$

As we have already shown for a fixed $B_n^\phi \in \mathcal{B}_n^\phi$ there are at most $C(K^2+2, X)$ balls $B_n \in \mathcal{B}_n$ such that $\phi(B_n)$ intersects B_n^ϕ. Therefore, in total we have

$$\#\{B_{n+1}^\phi \in \mathcal{I}_{n+1}^\phi \backslash \mathcal{I}_{n+1}^{n\phi} \ : \ B_{n+1}^\phi \in \mathcal{S}(B_n^\phi, R_n)\} \leqslant (C(K^2 + 2, X))^2 r_{n,n}.$$

We proceed further with the Cantor set construction by constructing the collection $I_{n+1}^{m\phi}$ from $I_{n+1}^{(m+1)\phi}$ $(0 \leqslant m < n)$ by removing all balls $B_{n+1}^\phi \in I_{n+1}^{(m+1)\phi}$ which have nonempty intersection with at least one of the sets $\phi(B_{n+1})$ where $B_{n+1} \in \mathcal{I}_{n+1}^{m+1} \backslash \mathcal{I}_{n+1}^m$. The same arguments as before yield for every ball $B_m^\phi \in \mathcal{B}_m^\phi$ the estimate

$$\#\left\{B_{n+1}^\phi \in \mathcal{I}_{n+1}^{(m+1)\phi} \backslash \mathcal{I}_{n+1}^{m\phi} \ : \ B_{n+1}^\phi \in \mathcal{S}\left(B_m^\phi, \prod_{i=0}^{n-m} R_i\right)\right\} \leqslant (C(K^2 + 2, X))^2 r_{m,n}.$$

This completes the proof that $\mathcal{K}$ is $(I\phi(B), \mathbf{R}, C\mathbf{r})$-Cantor with $C := (C(K^2 + 2, X))^2$. □

4 Cantor-winning sets

Theorems 3.1 – 3.3 show that under certain conditions on the sequences $\mathbf{R}$ and $\mathbf{r}_i$ the finite intersection

$$\bigcap_{i=1}^{k} \mathcal{K}(B, \mathbf{R}, \mathbf{r}_i)$$

is non-empty or even has positive Hausdorff dimension. However they do not cover countable intersections. Moreover, one can easily provide a finite collection of generalized Cantor sets on X which have empty intersection. Theorem 3.3* on the other hand states that under even stronger conditions we may deduce a similar statement for a countable intersection of Cantor sets. However, all of these conditions are somewhat cumbersome and may be quite difficult to check. The aim of this section is to define a collection of sets which satisfy properties (W1) and (W2) of winning sets (that is, their Hausdorff dimension equals to $\dim A_\infty(B)$ and their countable intersection has the same property) and whose qualifying conditions are much more clear cut.

Consider the constant sequence $\mathbf{R} = R, R, R, \ldots$. In this case we will denote any associated generalised Cantor set $\mathcal{K}(B, \mathbf{R}, \mathbf{r})$ (respectively local Cantor set $\mathcal{LK}(B, \mathbf{R}, \mathbf{s})$) by $\mathcal{K}(B, R, \mathbf{r})$ (respectively $\mathcal{LK}(B, R, \mathbf{s})$). An easy inspection of the construction algorithm for generalized Cantor sets on X precipitates the following proposition which will play a crucial role in constructing our new class of sets.

Proposition 4.1. *Let $R \in U$, $k \in \mathbb{N}$. Then $\mathcal{K}(B, R^k, \mathbf{r})$ is also $(B, R, \mathbf{t})$-Cantor where*

$$t_{m,n} := \begin{cases} r_{m/k,(n+1)/k-1} & \text{if } m \equiv n+1 \equiv 0 \pmod{k}; \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Now, we are prepared to give the formal definition of Cantor-winning set, the main object of interest in this paper.

Definition. *Fix a ball $B \in \mathcal{B}(X)$. Given a parameter $\epsilon_0 > 0$ we say a set $K \in X$ is **$\boldsymbol{\epsilon_0}$-Cantor-winning on $\mathbf{B}$ for the splitting structure** $(\mathbf{X}, \mathcal{S}, \mathbf{U}, \mathbf{f})$ if for every $0 < \epsilon < \epsilon_0$ there exists $R_\epsilon \in U$ such that for every $R \geqslant R_\epsilon$ with $R \in U$ the set K contains some $(B, R, \mathbf{r})$-Cantor set where*

$$r_{m,n} = f(R)^{(n-m+1)(1-\epsilon)} \quad \text{for every } m, n \in \mathbb{N}, m \leqslant n. \tag{20}$$

If the splitting structure $(X, \mathcal{S}, U, f)$ is fixed then for conciseness we omit its mention and simply say K is ϵ_0-*Cantor-winning on* B. Similarly, unless otherwise specified a set $K \subset \mathbb{R}^k$ or $K \subset \mathbb{Q}_p^k$ will be referred to as being ϵ_0-Cantor-winning on B if K is ϵ_0-Cantor-winning on B with respect to the relevant canonical splitting structure.

Definition. *If a set $K \in X$ is ϵ_0-Cantor-winning on B for every ball $B \in \mathcal{B}(X)$ then we say that K is **$\boldsymbol{\epsilon_0}$-Cantor-winning**, and simply **Cantor-winning** if K is ϵ_0-Cantor-winning for some $\epsilon_0 > 0$.*

We may apply Theorem 3.2 to estimate the Hausdorff dimension of Cantor-winning sets.

Theorem 4.1. *If the complete metric space X satisfies condition (S4), then for any $B \in \mathcal{B}(X)$ and any $\epsilon_0 > 0$ the Hausdorff dimension of an ϵ_0-Cantor-winning set on B is at least $\dim A_\infty(B)$.*

Proof. If the splitting structure $(X, \mathcal{S}, U, f)$ is trivial then $\dim \mathcal{K}(B, R, \mathbf{r}) = \dim A_\infty(B) = 0$. Otherwise, by taking if needed a power of R in place of R one can guarantee that $f(R) > 4$. Also, condition (12) is obviously satisfied. Also, in this case the final condition (13) condenses to the following:

$$\sum_{k=0}^{n} f(R)^{(k+1)(1-\epsilon)} \left(\frac{4}{f(R)}\right)^k \leqslant \frac{f(R)}{4}.$$

One can easily check that it is true for $f(R)$ large enough. So by again replacing R with a proper integer power of R if necessary we get that (13) is satisfied. Thus, for any Cantor-winning set E we have

$$\dim E \geqslant \dim A_\infty(B) - \log_R 2.$$

This estimate holds true with any integer power R^k in place of R, and the theorem is proven. □

Corollary. *Let K be a Cantor-winning set. Then, for any $B \in \mathcal{B}(X)$ we have*

$$\dim(K \cap A_\infty(B)) \;=\; \dim(A_\infty(B)).$$

In particular, in the case that $A_\infty(B) = B$ for every $B \in \mathcal{B}(X)$ we have $\dim(K) = \dim(X)$.

Next, we will show that the countable intersection of ϵ_0-Cantor-winning sets is again ϵ_0-Cantor-winning.

Theorem 4.2. *Let a splitting structure $(X, \mathcal{S}, U, f)$ be nontrivial. Then, given $\epsilon_0 > 0$ and a countable collection $\{K_i\}_{i\in\mathbb{N}}$ of ϵ_0-Cantor-winning sets, the intersection*

$$\bigcap_{i=1}^{\infty} K_i$$

is also ϵ_0-Cantor-winning.

Proof. Consider an arbitrary positive $\epsilon < \epsilon_0$. By the definition of ϵ_0-Cantor-winning sets we have that K_1 contains some $\mathcal{K}(B, R, \mathbf{r}_1)$ for R large enough where $r^{(1)}_{m,n} = f(R)^{(n-m+1)(1-\epsilon)}$. Choose R_ϵ such that $t \leqslant f(R_\epsilon)^{t(\epsilon_0-\epsilon)}$ for any positive integer t. Then for each $i > 1$ one can inductively find $k_i \in \mathbb{N}$ large enough such that $k_{i+1} > k_i$ and the set K_i contains a $\mathcal{K}(B, R_\epsilon^{k_i}, \mathbf{r}_i)$. Here $\mathbf{r}_i$ are defined by the formula (20):

$$r^{(i)}_{m,n} = f(R_\epsilon)^{k_i(n-m+1)(1-\epsilon)}.$$

By the definition of generalised Cantor sets any $(B, R_\epsilon, \mathbf{r}_i)$-Cantor set is also $(B, R_\epsilon, \tilde{\mathbf{r}}_i)$-Cantor as soon as $r^{(i)}_{m,n} \leqslant \tilde{r}^{(i)}_{m,n}$. Therefore, without loss of generality we can always assume that $\epsilon > \epsilon_0/2$. Next, we use Proposition 4.1 to deduce that K_i is also $(B, R_\epsilon, \mathbf{t}_i)$-Cantor, where $\mathbf{t}_i$ is computed from $\mathbf{r}_i$ by formula (19). This enables us to implement Theorem 3.3*, which yields that

$$\bigcap_{i=1}^{\infty} K_i \supset \bigcap_{i=1}^{\infty} \mathcal{K}(B, R_\epsilon, \mathbf{t}_i) = \mathcal{K}(B, R_\epsilon, \mathbf{t}),$$

where $t_{m,n} = \sum_{i=1}^{\infty} t^{(i)}_{m,n}$.

Finally, we must check that the values $t_{m,n}$ satisfy condition (20). Notice that $t^{(1)}_{m,n}$ always contributes the value $f(R_\epsilon)^{(n-m+1)(1-\epsilon)}$ to $t_{m,n}$. For $i > 1$ this contribution comprises

$$f(R_\epsilon)^{k_i((n+1)/k_i-1-m)/k_i+1)(1-\epsilon)} = f(R_\epsilon)^{(n-m+1)(1-\epsilon)}$$

if $m \equiv n+1 \equiv 0 \pmod{k_i}$. Otherwise, $t^{(i)}_{m,n}$ does not contribute anything to $t_{n,m}$. In other words, we have

$$t_{m,n} = f(R_\epsilon)^{(n-m+1)(1-\epsilon)} \cdot \#\{i \in \mathbb{N} \, : \, m \equiv n+1 \equiv 0 \pmod{k_i}\}. \tag{21}$$

Since all the numbers k_i are distinct the cardinality of the set on the right hand side of (21) is at most $n - m + 1$ and so we have

$$t_{m,n} \leqslant (n-m+1) f(R_\epsilon)^{(n-m+1)(1-\epsilon)} \leqslant f(R_\epsilon)^{(n-m+1)(1-2\epsilon+\epsilon_0)}.$$

The final inequality holds due to the choice of R_ϵ. As ϵ runs within the range $(\epsilon_0/2, \epsilon_0)$, the value $2\epsilon - \epsilon_0$ takes any value within $(0, \epsilon_0)$. Therefore, the intersection $\bigcap_{i=1}^\infty K_i$ contains a generalised Cantor set $\mathcal{K}(B, R_\epsilon, \mathbf{t})$ satisfying property (20). Finally, the same arguments apply if the parameter R_ϵ is replaced by any other value of $R \in U$ with $R > R_\epsilon$. This completes the proof. □

Theorem 4.3. *Let $(X, \mathcal{S}, U, f)$ be a splitting structure on a complete metric space X satisfying condition (S5). Assume also that $A_\infty(B) = B$ for each ball $B \in \mathcal{B}(X)$ and let $\phi \,:\, X \to X$ be a bi-Lipschitz homeomorphism. If $K \subset X$ is ϵ_0-Cantor-winning on a ball B then $\phi(K)$ is ϵ_0-Cantor-winning on $I\phi(B)$.*

Moreover if K is ϵ_0-Cantor-winning then so is its image $\phi(K)$.

Proof. It suffices to combine the definition of an ϵ_0-Cantor-winning set with Theorem 3.4. Indeed, consider an arbitrary $0 < \epsilon < \epsilon_0$. Then, by definition there exists $R_\epsilon \in U$ such that for $R \geq R_\epsilon$ the set K contains some $(B, R, \mathbf{r})$-Cantor set where $r_{m,n} = f(R)^{(n-m+1)(1-\epsilon)}$. By Theorem 3.4, the image $\phi(K)$ contains an $(I\phi(B), R, C\mathbf{r})$-Cantor set for some absolute positive constant C independent of R and ϵ. By choosing ϵ' satisfying $\epsilon < \epsilon' < \epsilon_0$ and $R_{\epsilon'}$ large enough so that $f(R_1)^{\epsilon'-\epsilon} > C$ it follows that for $R > \max\{R_\epsilon, R_{\epsilon'}\}$ one has $Cr_{m,n} \leqslant f(R)^{(n-m+1)(1-\epsilon')}$. Thus, the set $\phi(B)$ is ϵ_0-Cantor-winning on $I\phi(B)$.

To prove the final statement we take an arbitrary ball $B \in \mathcal{B}(X)$ and consider its preimage $\phi^{-1}(B)$. Take the circumscribed ball $E\phi^{-1}(B)$. Since K is ϵ_0-Cantor-winning it is in particular ϵ_0-Cantor-winning on $E\phi^{-1}(B)$. Therefore, the image $\phi(K)$ is ϵ_0-Cantor-winning on $I\phi(E\phi^{-1}(B))$. The final observation is that $I\phi(E\phi^{-1}(B)) = B$. This shows that $\phi(B)$ is indeed ϵ_0-Cantor-winning. □

Remark 4.1. In [9] the similar notion of *Cantor rich sets in* $\mathbb{R}$ was independently introduced. With some effort this concept could also be generalised to $\mathbb{R}^N$ and in turn arbitrary complete metric spaces. Cantor rich sets are also known to satisfy conditions (W1) and (W2). However, in the authors' opinion the conditions of ϵ_0-Cantor-winning sets are easier to check yet retain the same desirable properties. Furthermore, the following section provides some reasoning as to why our setup may be preferable in many cases (see Remark 5.1 and [4]). It would be interesting to compare the two notions, to ask whether the two concepts are equivalent, whether one of them includes another, or if neither of these two possibilities hold, although this appears to be a quite difficult and nuanced question.

5 Relationship with classical winning sets

We have shown that under certain conditions Cantor-winning sets satisfy the same desirable properties (W1) – (W3) as classical winning sets in $\mathbb{R}^N$. It is therefore natural to ask if and how these two concepts are compatible.

In his original paper, Schmidt defined his game in the context of any complete metric space X. For the (α, β)-game played on X, Alice and Bob pick successive nested balls in the same manner as described in Section 1.2. The definitions of α-winning sets and winning sets for gameplay in an arbitrary complete metric space are entirely analogous to those for the game played in $\mathbb{R}^N$. Strictly speaking, since a generic ball in X may not necessarily have a unique centre or radius, Alice and Bob should pick successive pairs of centres and radii satisfying some partial ordering as opposed to simply picking successive nested balls. However, for the sake of clarity one may simply assume that this nuance is accounted for in each of Alice's and Bob's strategies.

Properties (W2) and (W3) are satisfied by winning sets for any (α, β)-game played in an arbitrary complete metric space X (see [27] and [14] respectively). On the other hand, winning sets need not satisfy property (W1) in general. Indeed, Proposition 5.2 of [21] provides an example of a winning set of zero Hausdorff dimension. In [21] it is also shown that if X supports a measure satisfying certain desirable regularity conditions then property (W1) does indeed hold. We discuss one such property in a later section - see (24).

Comparing directly the property of being a winning set in X with the property of being a Cantor-winning in X appears to be a very difficult problem and would likely require lengthy and technical discussion. For this reason, and to help maintain the flow of this paper, we only mention that the authors intend to return to this topic in the subsequent work [4]. It is though much more feasible to place our framework within the hierarchy of various classes of games related to those of Schmidt that exhibit a slightly higher level of rigidity. Indeed, as we will see the relationship between Cantor-winning sets and the 'winning sets' of these classes of games is rather clear cut.

5.1 McMullen's Game

In [23], McMullen proposed the following one-parameter variant of Schmidt's game, defined in such a way that instead of choosing a region where Bob must play, Alice must now choose a region where he must not play. To be precise, first choose some parameter $\beta \in (0, \gamma(X))$, where $\gamma(X) > 0$ is some absolute constant (to be determined later) depending on the metric space X. McMullen's *β-absolute game* begins with Bob picking some initial ball $B_1 \in \mathcal{B}(X)$. Alice and Bob then take it in turns to place successive balls in such a way that

$$B_1 \supset B_1 \setminus A_1 \supset B_2 \supset B_2 \setminus A_2 \supset B_3 \supset \cdots,$$

subject to the conditions

$$\operatorname{rad}(B_{i+1}) \geq \beta \cdot \operatorname{rad}(B_i), \quad \operatorname{rad}(A_i) \leq \beta \cdot \operatorname{rad}(B_i), \quad \forall\, i \in \mathbb{N}.$$

We say a set $E \subset X$ is *β-absolute winning* if Alice has a strategy which guarantees

$$\bigcap_{i\in\mathbb{N}} B_i \cap E \neq \emptyset \tag{22}$$

for the game with parameter β. The set E is said to be *absolute winning* if it is β-absolute winning for every $\beta \in (0, \gamma(X))$. Note that in general $\bigcap_i B_i$ may not necessarily be a single point as in Schmidt's (α, β)-game.

McMullen's original definition of the β-absolute game exclusively involved the selection of closed balls in $\mathbb{R}^N$. However, the mechanics described above make sense when outlining the rules for play with (metric) balls in any complete metric space X.

The purpose of the upper bound $\gamma(X)$ for the choice of β, as introduced above, is to ensure that at every stage of a β-absolute game there is always a legal place for Bob to place his ball wherever Alice may have placed her preceding ball. For the game played on $X = \mathbb{R}^N$ with Euclidean balls one may take $\gamma(X) = 1/3$ as per McMullen's original definition. To see that this condition is necessary, notice that for $\beta \geq 1/3$ Alice may then at any stage choose her ball A_i to simply be the ball B_i scaled down by β. In doing so she would leave no possible choice of ball B_{i+1} satisfying $B_i \setminus A_i \supset B_{i+1}$. However, for $\beta < 1/3$ such a choice is always possible in $\mathbb{R}^N$.

For the game played on an arbitrary complete metric space such a constant $\gamma(X)$ need not exist. However, it was recently observed in [18] (see their Lemma 4.2, and also [25, 30]) that a sufficient condition for the existence of $\gamma(X)$ is that the metric space in question is *uniformly perfect*. Recall that for $0 < c < 1$ a metric space X is said to be *c-uniformly perfect* if for every metric ball $B(x, r) \neq X$ we have $B(x, r) \setminus B(x, cr) \neq \emptyset$, and is said to be *uniformly perfect* if it is c-uniformly perfect for some c. If the metric space X is uniformly perfect one may then take $\gamma(X) = c/5$, although it should be noted that this is not necessarily the optimal (largest possible) choice. It is easy to see that if X is endowed with a non-trivial splitting structure and X satisfies condition (S4) with constant $C(X)$ then X is indeed uniformly perfect and so in the setting of this paper McMullen's game is always well defined. In particular, one may take $c = u_0^{-1}$, where $u_0 \in U$ is the smallest natural number for which $f(u_0) > C(X)$.

It is well known that an absolute winning set in $\mathbb{R}^N$ is α-winning for every $\alpha \in (0, 1/2)$, and that for the game played on a c-uniformly perfect complete metric space an absolute winning set is α-winning for every $\alpha \in (0, c/5]$ - see [23] and [18] respectively. In both cases it can be shown that the countable intersection of β-absolute winning sets is again β-absolute winning, and that the image of an absolute winning set under a bi-Lipschitz homeomorphism is again absolute winning. Thus, absolute winning sets also satisfy properties (W2) and (W3). In fact, it is the case (see Proposition 4.3(v) of [18]) that absolute winning sets in any uniformly perfect complete metric space satisfy the following slightly stronger version of the latter property:

(W3*) The image of any absolute winning set under a quasisymmetric homeomorphism is again absolute winning.

As before, absolute winning sets do not in general satisfy condition (W1), although if X supports a measure satisfying (24) it follows that property (W1) does hold. See [18] and the references therein for further criteria.

The following theorem reveals that absolute winning sets have an extremely clear cut relationship with Cantor-winning sets. We delay the proof to a later subsection.

Theorem 5.1. *Assume a complete metric space X is endowed with a non-trivial splitting structure $(X, \mathcal{S}, U, f)$ and that condition (S4) holds with constant $C(X)$. If $E \subset X$ is absolute winning then E is 1-Cantor-winning.*

Remark 5.1. Since completion of this project, the authors (in collaboration with with Nesharim) [4] have been able to show that the converse statement is in fact true, at least in the case of $\mathbb{R}^N$ with canonical splitting structure. That is; remarkably, the property of being 1-Cantor-winning in $\mathbb{R}^N$ is in fact equivalent to the property of being absolutely winning!

5.2 The Hyperplane Absolute Winning game and its variants

In [11], a class of variants of McMullen's game was introduced, the so-called *k-dimensional absolute winning* games. The most commonly utilised of these games is the *hyperplane absolute winning* (or *HAW*) game. The class of games in [11] was specifically defined for play on subsets of $\mathbb{R}^N$ and relies upon the existence of an underlying vector space. For this reason, in order to discuss k-dimensional absolute winning games in the full setting of this paper we would first have to attach further structure to our complete metric space X. In particular, if so inclined one could define the games for subsets of some given Banach space, but since such an extension has not yet appeared in the literature we content ourselves with the setting of $\mathbb{R}^N$ (with metric $\mathbf{d}(x,y) = |x-y|_\infty$ and canonical splitting structure) for the sake of clarity. Accordingly, we will refer to the metric balls in $\mathbb{R}^N$ as 'boxes'. That said, one should observe that an analogous method to the one we shall exhibit would be applicable to questions concerning k-dimensional absolute winning games played on more exotic spaces.

Firstly, fix $k \in \{0, 1, \ldots, N-1\}$ and some parameter $0 < \beta < 1/3$. The k-dimensional β-absolute winning game has the same premise as McMullen's game in that Alice must choose a region where Bob must not play, only now that region is defined by the neighbourhood of a k-dimensional affine subspace of $\mathbb{R}^N$ rather than the neighbourhood of a single point. The game begins with Bob picking some initial box $B_1 \subset \mathbb{R}^N$. Now, assume Bob has played his i-th box B_i. Then the game proceeds with Alice choosing $\delta_i \leqslant \beta$ and an affine subspace $\mathcal{L}_i$ of dimension k and removing its $(\delta_i \cdot \mathrm{rad}(B_i))$-neighbourhood

$$\mathcal{L}_i^{(\delta_i \cdot \mathrm{rad}(B_i))} = \left\{ x \in \mathbb{R}^N : \inf_{y \in \mathcal{L}_i} |x-y|_\infty < \delta_i \cdot \mathrm{rad}(B_i) \right\}$$

from the box B_i. In accordance with this procedure, set $A_i := \mathcal{L}_i^{(\delta_i \cdot \mathrm{rad}(B_i))} \cap B_i$. Then, Bob for his $(i+1)$-th move may choose any box $B_{i+1} \subset B_i \backslash A_i$ satisfying $\mathrm{rad}(B_{i+1}) \geq \beta \cdot \mathrm{rad}(B_i)$. A set $E \subset \mathbb{R}^N$ is said to be *k-dimensionally β-absolute winning* if Alice has a strategy guaranteeing that

$$\bigcap_{i=1}^{\infty} B_i \cap E \neq \emptyset$$

for the game played with parameter β. We simply say that E is *k-dimensionally absolute winning* if it is *k-dimensionally β-absolute winning* for every $\beta \in (0, 1/3]$.

In the weakest case '$k = N-1$', the game is often referred to as the *hyperplane absolute winning* game for obvious reasons. For simplicity, an $(N-1)$-dimensionally absolute winning set is then referred to as being *hyperplane absolute winning (HAW)*. One can readily observe that the strongest case '$k = 0$' coincides with McMullen's original game on $\mathbb{R}^N$. For every k, if a set is k-dimensionally absolute winning then it is α-winning with respect to Schmidt's game for any $\alpha \in (0, 1/2)$. We direct the reader to [11] for further discussion of the properties of k-dimensionally absolute winning sets.

Theorem 5.2. *Assume a subset $E \subset \mathbb{R}^N$ is k-dimensionally absolute winning for some integer $k \in \{0, 1, \ldots, N-1\}$. Then, the set E is $\frac{N-k}{N}$-Cantor-winning.*

Note that the case '$k = 0$' corresponding to McMullen's game is contained within the statement of Theorem 5.1. Broadly speaking, Theorems 5.1 & 5.2 demonstrate that the property of being a Cantor-winning set is weaker than the property of being a k-dimensionally absolute winning set (for any given $k \in \{0, 1, \ldots, N-1\}$).

5.3 Proof of Theorems 5.1 & 5.2

5.3.1 Preliminaries

In order to present our proofs we first require some terminology. For consistency we use the notation originally introduced in [27]. Additionally, for $k = 0, 1, \ldots, N-1$ let $\mathcal{H}_k$ denote the set of all k-dimensional affine subspaces in $\mathbb{R}^N$.

In each of the k-dimensionally absolute winning games (including McMullen's game on a metric space X), a set E is (k-dimensionally) absolute winning if Alice has a 'strategy' for placing her moves A_i so that however Bob chooses to place his balls B_i the set $\bigcap_i B_i$ intersects E. Formally, a *strategy* $F := (f_1, f_2, \ldots)$ is a sequence of functions $f_i : \mathcal{B}(X)^i \to \mathcal{H}_k \times \mathbb{R}_{>0}$. Given a fixed parameter β, we say a strategy F is *legal* for the (k-dimensional) β-absolute game if it satisfies the following property for any finite sequence $(b_1, \ldots, b_n)$ of balls, any affine subspace h, and any $s \in \mathbb{R}_{>0}$:

$$\text{if } f_n(b_1, \ldots, b_n) = (h, s), \text{ then } s \leq \beta \cdot \mathrm{rad}(b_n). \tag{23}$$

Similarly, an individual move by Alice or Bob is said to be *legal* if it satisfies the rules of the game outlined earlier. For $(h, s) \in \mathcal{H}_k \times \mathbb{R}_{>0}$ denote by $g(h, s) := h^{(s)}$ the standard closed s-neighbourhood of the subspace h. We say F is a *winning strategy (for E)* with respect to the game with parameter β if firstly it is legal and secondly it then determines where Alice should place her moves $A_i := g(f_i(B_1, B_2, \ldots, B_i))$ in such a way that, however we choose to place Bob's balls B_1 and $B_{i+1} \subset B_i \setminus A_i$ (for $i \in \mathbb{N}$) legally in the game, condition (22) holds. In this notation, a set E is (k-dimensionally) β-absolute winning if and only if there exists a winning strategy for E with respect to the (k-dimensionally) β-absolute game.

The following key observation made by Schmidt in [27] (his Theorem 7) allows us to significantly simplify our notation: In any of the above games, the existence of a winning strategy for a set E guarantees the existence of a 'positional' winning strategy for E. We say a winning strategy $F := (f_1, f_2, \ldots)$ is *positional* if there is a global function $f_0 : \mathcal{B}(X) \to \mathcal{H}_k \times \mathbb{R}_{>0}$, independent of i, for which each function f_i satisfies $f_i(B_1, \ldots, B_i) = f_0(B_i)$; that is, the placement of each of Alice's moves in the winning strategy depends only upon the position of Bob's immediately preceding ball, not on the entirety of the game so far. For this reason, if the ball b appears as Bob's n-th move during gameplay then we will without loss of generality write $g(f_0(b))$ to denote Alice's subsequent move as determined by a positional strategy F and if clarity is required write $F = F(f_0)$.

As a final piece of terminology from [27], given a target set E we refer to a sequence $(B_1, B_2, \ldots)$ of balls as an *F-chain* if it consists of the moves Bob has made during a (k-dimensional) β-absolute game in which Alice has followed the winning strategy F for E. By definition we must have that (22) holds for this sequence. Furthermore, we say a finite sequence $(B_1, B_2, \ldots B_n)$ is an *F_n-chain* if there exist $B_{n+1}, B_{n+2}, \ldots$ for which the infinite sequence $(B_1, B_2, \ldots B_n, B_{n+1} \ldots)$ is an F-chain.

5.3.2 Proof of Theorem 5.1

Recall that for any non-trivial splitting structure whose underlying metric space X satisfies condition (S4) the quantity $u_0 \in U$ is defined to be the smallest number such that $f(u_0) > C(X)$. By assumption our set $E \subset X$ is β-absolute winning for every $\beta < \gamma(X) := (5u_0)^{-1}$. Fix some ball $B \in \mathcal{B}(X)$ and $\epsilon \in (0, 1)$, and let $R_1 \in U$ be the smallest integer for which $5u_0 < R_1$. Next, choose $R_2 \in U$ large enough so that for any $R \in U$ with $R \geq R_2$ we have $f(R)^{(1-\epsilon)} \geq C(X)$. This is always possible for a non-trivial splitting structure by Corollary 2.4 and the multiplicativity of f. Now set $R_\epsilon := \max(R_1, R_2)$. To prove the theorem it suffices to construct for each $R \in U$ with $R \geq R_\epsilon$ a local Cantor set $\mathcal{LK}(B, R, \mathbf{s})$ lying inside E for which $s_n \leq f(R)^{(1-\epsilon)}$.

Fix some $R \in U$ satisfying $R \geq R_\epsilon$. Our method for constructing the set $\mathcal{LK}(B, R, \mathbf{s})$ is as follows. We play as Bob in an iteration of McMullen's game with parameter $\beta = 1/R$. By assumption the set E is $(1/R)$-absolute winning and so there exists a winning strategy for E with respect to this game. Schmidt's result implies that there then must exist a positional winning strategy $F(f_0)$, which we will enforce Alice to adhere to. Note, we have $\mathcal{H}_0 = X$ here and so $\{g(h,s) : (h,s) \in X \times \mathbb{R}_{>0}\}$ coincides with the set of all closed balls $\mathcal{B}(X)$ in X.

Assume that Bob plays his first ball in position $B_1 = B$ and allow the positional strategy F to determine Alice's first ball $A_1 := g(f_0(B))$. Consider the set $\mathcal{S}(B,R)$. Since by (23) we have $\mathrm{rad}(A_1) \le \frac{1}{R}\mathrm{rad}(B) = \mathrm{rad}(b)$ for every $b \in \mathcal{S}(B,R)$, the ball A_1 may intersect at most $C(X)$ balls from the collection $\mathcal{S}(B,R)$.

The construction of the local Cantor set $\mathcal{LK}(B,R,\mathbf{s})$ comprises the construction of sub-collections $\mathcal{B}_i \subset \mathcal{S}(B,R^i)$ and a sequence $\mathbf{s} = (s_n)_{n\in\mathbb{Z}_{\geqslant 0}}$. As a first step in this procedure, define $\mathcal{B}_0 := \{B\}$ and

$$\mathcal{B}_1 := \{b \in \mathcal{S}(B,R) : \; g(f_0(B)) \cap b = \emptyset\}.$$

Upon setting $s_0 := \#(\mathcal{S}(B,R) \setminus \mathcal{B}_1)$ we have $s_0 \le C(X) \le f(R)^{(1-\epsilon)}$ as required. Furthermore, any ball $B_2 \in \mathcal{B}_1$ is a legal choice for Bob's next move in the game; i.e., the finite sequence (B, B_2) is an F_2-chain for any $B_2 \in \mathcal{B}_1$.

Assume now that for some $n \in \mathbb{N}$ we have constructed the collections $\mathcal{B}_i$ and defined the values s_{i-1} for $i = 1,\ldots,n$. Assume also that these collections satisfy the property that for every $b \in \mathcal{B}_i$ we have $g(f_0(b')) \cap b = \emptyset$, where b' is the unique ball in the collection $\mathcal{B}_{i-1}$ containing b. It is immediate that any finite sequence $(B_1,\ldots B_{n+1})$ with $B_i \in \mathcal{S}(\mathcal{B}_{i-1},R)$ (when $i > 1$) is an F_{n+1}-chain. We construct the collection $\mathcal{B}_{n+1}$ in the following way. Simply notice that for any $b' \in \mathcal{B}_n$ the ball $g(f_0(b'))$ may, by (23) and condition (S4), intersect at most $C(X)$ of the balls from the collection $S(b',R)$. Indeed, for $b' \in \mathcal{B}_n$ let

$$\mathcal{B}_{n+1}^{b'} := \left\{b \in \mathcal{S}(b',R) : \;\; g(f_0(b')) \cap b = \emptyset\right\},$$

and set

$$\mathcal{B}_{n+1} := \bigcup_{b'\in\mathcal{B}_n} \mathcal{B}_{n+1}^{b'} \qquad \text{and} \qquad s_n := \max_{b'\in\mathcal{B}_n} \# \left(\mathcal{S}(b',R) \setminus \mathcal{B}_{n+1}^{b'}\right).$$

Then, it follows that $s_n \le C(X) \le f(R)^{(1-\epsilon)}$ and by definition that for every $b \in \mathcal{B}_{n+1}$ we have $g(f_0(b')) \cap b = \emptyset$, where b' is the unique ball in the collection $\mathcal{B}_n$ containing b. Furthermore, if $(B_1,\ldots B_{n+1})$ with $B_i \in \mathcal{B}_{i-1}$ is an F_{n+1}-chain then $(B_1,\ldots B_{n+1},b)$ is an F_{n+2}-chain for any $b \in \mathcal{B}_{n+1}$. This completes the inductive procedure.

Upon defining

$$\mathcal{LK}(B,R,\mathbf{s}) := \bigcap_{i\in\mathbb{Z}_{\geq 0}} \bigcup_{b\in\mathcal{B}_i} b,$$

it only remains to show that $\mathcal{LK}(B,R,\mathbf{s}) \subseteq E$. With this in mind, choose some point $x \in \mathcal{LK}(B,R,\mathbf{s})$ and let $s = (b_i)_{i\in\mathbb{N}}$ with $b_i \in \mathcal{B}_{i-1}$ be a sequence of balls for which $\bigcap_{i\in\mathbb{N}} b_i = \{x\}$. By construction, we have ensured that each finite subsequence $(b_1,\ldots,b_n)$ of s is an F_n-chain. Moreover, it is readily verified that if $(b_1,b_2,\ldots)$ is a sequence of balls such that for every $n \in \mathbb{N}$ the finite sequence $(b_1,\ldots b_n)$ is an F_n-chain, then $(b_1,b_2,\ldots)$ is an F-chain (c.f. [27, Lemma 1]). It follows that condition (22) holds and, since $\mathrm{rad}(b_i) \to 0$ as $i \to \infty$ implies the intersection $\bigcap_{i\in\mathbb{N}} b_i$ contains the single point x, that $x \in E$ as required.

Finally, by the fact that the initial ball B and the quantity $\epsilon \in (0,1)$ were arbitrary it follows that the set E is 1-Cantor-winning.

5.3.3 Proof of Theorem 5.2

The proof follows very similar arguments to those in the proof of Theorem 5.1. For this reason we only outline the modifications required. The key observation is that, given any box B of sidelength diam(B) in $\mathbb{R}^N$ and any $R \in \mathbb{N}$, the rectangular neighbourhood

$$\mathcal{L}^{(\mathrm{rad}(B)/R)} = \left\{ x \in \mathbb{R}^N : \inf_{y \in \mathcal{L}} |x - y|_\infty < \frac{\mathrm{rad}(B)}{R} \right\}$$

of any k-dimensional affine subspace $\mathcal{L}$ may intersect at most $c(k, N) \cdot R^k$ of the boxes $b \in S(B, R)$. Here, the quantity $c(k, N) \in \mathbb{R}_{>0}$ is an absolute constant depending only upon k and N.

Suppose the set $E \subset \mathbb{R}^N$ is k-dimensionally absolute winning. Fix some box $B \subset \mathbb{R}^N$ and some $\epsilon \in (0, (N - k)/N)$. Next, choose $R_\epsilon > 3$ large enough so that for any $R \geq R_\epsilon$ we have $R^{N(1-\epsilon)} \geq c(k, N) \cdot R^k$. This is always possibly since

$$1 - \epsilon \ > \ 1 - \frac{N - k}{N} \ = \ \frac{k}{N}.$$

Fix some $R \geq R_\epsilon$ and consider a β-absolute game with $\beta = 1/R$. By assumption there exists a winning strategy, and therefore a positional winning strategy $F(f_0)$, associated with the parameter β and the set E. Let Bob initially play the box $B_1 := B$ and set $\mathcal{B}_0 := \{B\}$. As in the proof of Theorem 5.1 one must construct collections $\mathcal{B}_n \in S(B, R^n)$ and a sequence $\mathbf{s} = (s_i)_{i \in \mathbb{N}}$ in an iterative fashion in order to define some local Cantor set $\mathcal{LK}(B, R, \mathbf{s})$. Given $n \geq 0$, for every box $b \in \mathcal{B}_n$ played by Bob during gameplay the strategy F determines the position and a neighbourhood of a k-dimensional affine subspace instructing Alice where to play her next move. By the above observation any such neighbourhood may intersect at most $c(k, N) \cdot R^k$ boxes from the collection $S(b, R)$. Following exactly the method of Theorem 5.1 one may analogously construct the required collections

$$\mathcal{B}_{n+1}^{b'} := \{ b \in \mathcal{S}(b', R) \ : \ g(f_0(b')) \cap b = \emptyset \} \quad \text{for } b' \in \mathcal{B}_n \quad \text{and} \quad \mathcal{B}_{n+1} := \bigcup_{b' \in \mathcal{B}_n} \mathcal{B}_n^{b'}.$$

Furthermore, we may choose $s_n := c(k, N) \cdot R^k \leq R^{N(1-\epsilon)}$ as required. As before, since F is a winning strategy it is ensured that the resulting local $(B, R, \mathbf{s})$-Cantor set falls inside E.

6 Generalized badly approximable sets

In [22] the authors introduced a broad notion of badly approximable sets. We now discuss how their setup is related to ours, and more importantly, how we are able to generalise their results. In the process we will outline an algorithm describing precisely how to apply our framework to any given 'badly approximable' set appearing in the study of Diophantine approximation. We begin by giving a brief outline of the framework outlined in [22], tailored to our needs.

Let X be a complete metric space (with metric $\mathbf{d}$) and let $\mathcal{R}$ be a family of subsets $\mathcal{R} := \{R_\alpha \subset X \ : \ \alpha \in S\}$ indexed by an infinite countable set S. In most of the applications we will discuss, the subsets R_α will consist simply of points in X. The subsets R_α will be referred to as *resonant sets*. We attach a 'weight' to each resonant set by introducing a function $h \ : \ S \to \mathbb{R}_{\geqslant 0}$. For convenience we will always assume that h is bounded above by some absolute constant; in other words, there exists a constant $C > 0$ such that for every $\alpha \in S$ we have $h(\alpha) \leqslant C$. Next, for any set $R \subset X$, let

$$\Delta(R, \delta) := \{ \mathbf{x} \in X \ : \ \mathbf{d}(\mathbf{x}, R) \leqslant \delta \}$$

denote the δ-neighbourhood of R. Finally, we say a set of the form

$$\mathbf{Bad}(\mathcal{R}, h) := \{ \mathbf{x} \in X \ : \ \exists c > 0, \forall \alpha \in S, x \notin \Delta(R_\alpha, c \cdot h(\alpha)) \}.$$

is a *generalised bad set*.

Remark 6.1. Our definition slightly differs to that given in [22] and is not quite as broad in scope. For simplicity we have combined the two functions β_α and ρ present in [22] into one function h. With reference to the notation of [22], if we take $\beta_\alpha = (h(\alpha))^{-1}$, $\rho(x) = x^{-1}$ and $\Omega = X$ then $\mathbf{Bad}^*(\mathcal{R}, \beta, \rho)$ defined in [22] is precisely our set $\mathbf{Bad}(\mathcal{R}, h)$. Furthermore, since there is a bijection between S and our subsets $\mathcal{R}$, in applications we will often use the notation $h(R_\alpha)$ for $R_\alpha \in \mathcal{R}$ instead of $h(\alpha)$ if it is convenient to do so.

The following all provide basic examples of generalised bad sets:

- The standard set **Bad** of badly approximable numbers. In this case an easy inspection shows that $\mathbf{Bad} = \mathbf{Bad}(\mathcal{R}, h)$ where $\mathcal{R}$ consists of rational points and $h(p/q) := 1/q^2$ for every $p \in \mathbb{Z}, q \in \mathbb{N}, \gcd(p, q) = 1$.

- The set $\mathbf{Bad}_N = \mathbf{Bad}(1/N, 1/N, \ldots, 1/N)$ of badly approximable points in $\mathbb{R}^N$ defined in (1). Again one can check that $\mathbf{Bad}_N$ is a generalised bad set for

$$\mathcal{R} = \{\mathbf{p}/q \ : \ \mathbf{p} \in \mathbb{Z}^N, q \in \mathbb{N}, \gcd(p_1, \ldots, p_N, q) = 1\}$$

  and $h(\mathbf{p}/q) = q^{-1-1/N}$.

- The set $\mathbf{Bad}_p := \mathbf{Bad}_p(1)$ of *p-adically badly approximable numbers* as defined in (6). The inequality in the definition of $\mathbf{Bad}_p$ is clearly satisfied for $q = 0$. We can also without loss of generality assume that $\gcd(q, r) = 1$. Then, by dividing both sides of the inequality in (6) by $|q|_p$ one can check that it is a generalised bad set for $\mathcal{R} = \mathbb{Q} \subset \mathbb{Q}_p$ and

$$h(r/q) = (\max\{|r|^2, |q|^2\} \cdot |q|_p)^{-1}.$$

In [22] the authors give quite general conditions on $\mathcal{R}$ and h which guarantee that a generalised bad set $\mathbf{Bad}(\mathcal{R}, h)$ has full Hausdorff dimension. Namely they prove the following.

Theorem KTV (Theorem 1 in [22])**.** *Let X support a measure m for which there exist strictly positive constants δ, r_0, a and b such that $a \leqslant 1 \leqslant b$ and for any $x \in X$ and $r \leqslant r_0$,*

$$ar^\delta \leqslant m(B(x, r)) \leqslant br^\delta. \tag{24}$$

Choose a sufficiently large natural number R and let $S(n) := \{\alpha \in S \ : \ R^{n-1} \leqslant (h(\alpha))^{-1} < R^n\}$. Assume that there exists $\theta \in \mathbb{R}^+$ so that for $n \geqslant 1$ and any ball $B_n = B(x, R^{-n})$ there exists a collection $C(\theta B_n)$ of disjoint balls such that

$$\forall B_{n+1} \in C(\theta B_n), \ \mathrm{rad}(B_{n+1}) = 2\theta R^{-(n+1)} \ \ \textit{and} \ \ B_{n+1} \subset B(x, \theta R^{-n});$$

$$\#C(\theta B_n) \geqslant \kappa_1 R^\delta$$

and

$$\#\{B_{n+1} \in C(\theta B_n) \ : \ \exists \alpha \in S(n+1) \ \textit{s.t.} \ \mathrm{cent}(B_{n+1}) \in \Delta(R_\alpha, 2\theta R^{-(n+1)})\} \leqslant \kappa_2 R^\delta,$$

where $0 < \kappa_2 < \kappa_1$ are absolute constants independent of k and n. Furthermore, suppose that $\dim(\cup_{\alpha \in J} R_\alpha) < \delta$. Then, $\dim(\mathcal{R}, h) = \delta = \dim X$.

Remark 6.2. Property (24) is usually referred to as *δ-Ahlfors regularity.*

This theorem provides the Hausdorff dimension for $\mathbf{Bad}(\mathcal{R}, h)$ in a wide ranging setup in which relatively mild (but rather technical) conditions on $\mathcal{R}$, X and m are assumed. However, we show that some sets $\mathbf{Bad}(\mathcal{R}, h)$ which do not satisfy certain conditions of Theorem KTV can still be shown to fall into a class of sets satisfying properties (W1) – (W3), as can many sets which do fall within the scope of [22]. In this sense our framework is more far reaching than that presented in [22]. On the other hand, in order to do this we will need to impose slightly more structure on the balls B_n and classes $C(\theta B_n)$, which in turn makes some of our conditions slightly stronger than those imposed in Theorem KTV.

Observe that one may consider the sets $\mathbf{Bad}(\mathcal{R}, h)$ as the set of points surviving after the removal of every neighborhood $\Delta(R_\alpha, c \cdot h(\alpha))$ from X. On adopting this point of view one may appreciate the similarity between general bad sets and generalized Cantor sets. To further illustrate this connection we now provide an algorithm, which will be referred as a *bad to Cantor set construction*, demonstrating that the intersection of every generalised bad set $\mathbf{Bad}(\mathcal{R}, h)$ with any set $A_\infty(B)$ contains some generalized Cantor set $\mathcal{K}(B, R, \mathbf{r})$.

Bad to Cantor Set Construction:

1. Fix R large enough and choose c small enough such that
$$\sup_{\alpha}\{c \cdot h(\alpha)\} \leqslant \operatorname{diam}(B) \cdot R^{-1}. \tag{25}$$
This can be done since $h(\alpha)$ is always bounded above by an absolute constant.

2. Split the collection $\mathcal{R}$ into classes $C(n)$, for $n \in \mathbb{N}$, in the following way. Let
$$C(n) := \{R_\alpha \in \mathcal{R} \ : \ \operatorname{diam}(B)R^{-n-1} < c \cdot h(\alpha) \leqslant \operatorname{diam}(B)R^{-n}\}. \tag{26}$$

3. Define $K_0 := \{B\}$. This constitutes the 0'th layer for the generalized Cantor construction.

4. On step n ($n \in \mathbb{N}$) we start with a collection K_{n-1} of balls. Define
$$L_n := \bigcup_{b \in K_{n-1}} \mathcal{S}(b, R).$$
Then, remove every ball from L_n that intersects $\Delta(R_\alpha, c \cdot h(\alpha))$ for at least one $R_\alpha \in C(n)$. Denote by K_n the collection balls that survive.

5. Finally, construct
$$K_\infty = K_\infty(R) := \bigcap_{n=0}^{\infty} \bigcup_{B \in K_n} B.$$

By the construction K_∞ is surely $(B, R, \mathbf{r})$-Cantor for some parameter $\mathbf{r}$. At the moment we do not have any restrictions on the values of $\mathbf{r}$, so theoretically $r_{n,n}$ could equal $f(R)$ and $K_\infty = \emptyset$. We must impose some conditions on a pair $(\mathcal{R}, h)$ in order to produce non-trivial generalized Cantor sets. Note that K_∞ can be constructed for all (sufficiently large) values R.

Assume next that every class $C(n)$ can be further split into subclasses $C(n, m)$, $1 \leqslant m \leqslant n$ such that for every ball $b \in K_{n-m}$ we have
$$\#\{D \in \mathcal{S}(b, R^m) \cap L_n \ : \ \exists R_\alpha \in C(n, m),\ D \cap \Delta(R_\alpha, c \cdot h(\alpha)) \neq \emptyset\} \ll f(R)^{m(1-\epsilon_0)}, \tag{27}$$
where $0 < \epsilon_0 < 1$ is some absolute constant. Then, one can make Step 4 of the above algorithm more specific:

4.1. Remove every ball from L_n which intersects with $\Delta(R_\alpha, c \cdot h(\alpha))$ for at least one $R_\alpha \in C(n, 1)$. By (27) it will remove at most $C_1 f(R)^{1-\epsilon_0}$ balls from each set $\mathcal{S}(b, R)$, $b \in K_{n-1}$. Here C_1 is some absolute positive constant.

4.m. (for $1 < m \leqslant n$). In general, for each $m \in \{2, \ldots, n\}$ remove every ball from L_n that intersects $\Delta(R_\alpha, c \cdot h(\alpha))$ for at least one $R_\alpha \in C(n,m)$. By (27) this process will remove at most $C_1 f(R)^{m(1-\epsilon_0)}$ balls from each set $\mathcal{S}(b, R^m)$, $b \in K_{n-m}$.

This updated procedure ensures that K_∞ is a $(B, R, \mathbf{r})$-Cantor set with $r_{m,n}$ satisfying (20) for every $\epsilon < \epsilon_0$ and R large enough.

Finally, we establish that each set $K_\infty(R)$ produced using the bad to Cantor construction lies inside $\mathbf{Bad}(\mathcal{R}, h)$. By the construction of each K_n we have

$$\Delta(R_\alpha, c \cdot h(\alpha)) \quad \cap \quad \bigcap_{n=0}^{m} \bigcup_{B \in K_n} B \; = \; \emptyset$$

for every $R_\alpha \in \bigcup_{n=1}^{m} C(n)$. By letting m tend to infinity we find $K_\infty(R) \subset \mathbf{Bad}(\mathcal{R}, h)$. In other words for each R large enough there exists a $(B, R, \mathbf{r})$-Cantor subset $K_\infty(R)$ of $\mathbf{Bad}(\mathcal{R}, h)$ with $r_{m,n}$ satisfying (20). This in turn implies that $\mathbf{Bad}(\mathcal{R}, h)$ is ϵ_0-Cantor winning.

To summarize, we have proved the following theorem.

Theorem 6.1. *Let $(X, \mathcal{S}, U, f)$ be a splitting structure on X and $\mathbf{Bad}(\mathcal{R}, h) \subset X$ be a generalized bad set. If $\mathbf{Bad}(\mathcal{R}, h)$ adopts a bad to Cantor set construction with condition* (27) *satisfied for some $\epsilon_0 > 0$ and some $B \in \mathcal{B}(x)$ then it is ϵ_0-Cantor-winning on B. In particular if X satisfies property (S4) then*

$$\dim(\mathbf{Bad}(\mathcal{R}, h) \cap A_\infty(B)) = \dim A_\infty(B).$$

Moreover, if the former conditions are satisfied for any ball $B \in \mathcal{B}(X)$ and fixed $\epsilon_0 > 0$ then $\mathbf{Bad}(\mathcal{R}, h)$ is ϵ_0-Cantor-winning. If additionally X satisfies Property (S5) and for any $B \in \mathcal{B}(x)$ we have $A_\infty(B) = B$ then for any bi-Lipschitz homeomorphism $\phi : X \to X$, $\phi(\mathbf{Bad}(\mathcal{R}, h))$ is also ϵ_0-Cantor-winning.

Theorem 6.1 is in some sense quite cumbersome. One needs to go through the whole bad to Cantor set construction in order to check its conditions. In particular, one needs to construct the sets K_n and L_n. However, via a minor sacrifice in generality one can improve the accessibility of Theorem 6.1 and make it independent of any particular bad to Cantor set construction. Moreover, one can ensure the conditions are independent on the particular splitting structure. This potentially provides the means to simultaneously establish a Cantor-winning property of a set $\mathbf{Bad}(\mathcal{R}, h)$ for various splitting structures $(X, \mathcal{S}, U, f)$ of the metric space X.

We first require some further notation. For some R and c satisfying (25), assume we are given a class $C(n)$ defined by (26) and a collection of subclasses $C(n,m)$ for $1 \leqslant m \leqslant n$. For any ball $b \in \mathcal{B}(X)$ let $q_{n,m}(b)$ denote the maximum number of balls $D \subset b$ of radius $\mathrm{rad}(b)R^{-m}$ such that they may intersect only on their boundaries and there exists $R_\alpha \in C(n,m)$ satisfying $D \cap \Delta(R_\alpha, c \cdot h(\alpha)) \neq \emptyset$. Then, define

$$q_{n,m} := \sup\{q_{n,m}(b) \; : \; b \in \mathcal{B}(X), \mathrm{rad}(b) = \mathrm{rad}(B)R^{m-n}\}.$$

We may now introduce our simplification of Theorem 6.1.

Corollary 6.2. *Fix $B \in \mathcal{B}(X)$ and let the parameters R and c satisfy* (25)*. Also, assume that for $n \in \mathbb{N}$ we have classes $C(n)$ defined by* (26)*, each associated with a collection of subclasses $C(n,m)$ for $1 \leqslant m \leqslant n$. If for all pairs m, n and for some $\epsilon_0 > 0$ a splitting* *structure $(X, \mathcal{S}, U, f)$ satisfies $q_{n,m} \ll R^{m(1-\epsilon_0)}$, then* $\mathbf{Bad}(\mathcal{R}, h)$ *is ϵ_0-Cantor-winning on B with respect to $(X, \mathcal{S}, U, f)$. In particular, we have*

$$\dim(\mathbf{Bad}(\mathcal{R}, h) \cap A_\infty(B)) = \dim A_\infty(B).$$

Proof. We must simply apply the bad to Cantor set construction. Every ball b in K_{n-m} has radius $\mathrm{rad}(b) = \mathrm{rad}(B) \cdot R^{m-n}$ and all balls in $\mathcal{S}(b, R^m) \cap L_n$ are disjoint and have radius $\mathrm{rad}(b)R^{-m}$. Therefore, the expression on the l.h.s. of (27) does not exceed $q_{n,m}(b)$. In turn, this is at most $q_{n,m}$. Finally, the inequality $q_{n,m} \leqslant R^{m(1-\epsilon_0)}$ assures that (27) is satisfied and so Theorem 6.1 can be readily applied. □

Simplified Bad to Cantor-Winning Procedure:

As a conclusion, Theorem 6.1 and its corollary suggest the following procedure to check the Cantor-winning property for a given set $\mathbf{Bad}(\mathcal{R}, h)$.

- Take any large enough $R \in \mathbb{N}$ and a small fixed $c > 0$ in such a way that (25) is satisfied.
- Construct the classes $C(n)$ defined by (26). This constitutes a more or less straightforward task. Then, split each $C(n)$ into suitable subclasses $C(n, m)$. This often proves trickier. For 'classical' sets $\mathbf{Bad}(\mathcal{R}, h)$ it is often sufficient to take $C(n, 1) := C(n)$ and $C(n, m) := \emptyset$ for $m \geqslant 2$. However, for various more 'modern' badly approximable sets more care is needed in the dividing process.
- Compute an upper estimate for $q_{n,m}$; i.e., for each small ball b of radius $\mathrm{rad}(B)R^{-n+m}$ consider the set
$$\bigcup_{R_\alpha \in C(n,m)} \Delta(R_\alpha, c \cdot h(\alpha)) \cap b$$
and estimate the number of disjoint balls of smaller radius $\mathrm{rad}(B)R^{-n}$ that may intersect this set.
- If this estimate is tight enough so that $q_{n,m} \ll f(R)^{m(1-\epsilon_0)}$ then for the splitting structure $(X, \mathcal{S}, U, f)$ the set $\mathbf{Bad}(\mathcal{R}, h)$ is ϵ_0-Cantor-winning.

We give examples of how this procedure may be implemented for various badly approximable sets in the next section. Beforehand, we conclude this section with a treatment of the special case that $\mathcal{R}$ consists only of points. Here, every $\Delta(R_\alpha, c \cdot h(\alpha))$ is simply a ball of radius $c \cdot h(\alpha)$ and the conditions one must check to establish the Cantor-winning property of a set become even simpler.

By the definition of the class $C(n)$, for each $R_\alpha \in C(n)$ and for each ball D of radius $\mathrm{rad}(D) = \mathrm{rad}(B)R^{-n}$ one has

$$\mathrm{rad}(\Delta(R_\alpha, c \cdot h(\alpha))) \leqslant \mathrm{rad}(D).$$

Therefore, if X satisfies condition (S4) then the ball $\Delta(R_\alpha, c \cdot h(\alpha))$ can intersect at most $C(X)$ such disjoint balls D of radius $\mathrm{rad}(B)R^{-n}$. Hence we have $q_{n,m}(b) \leqslant C(X)\tilde{q}_{n,m}(b)$ where

$$\tilde{q}_{n,m}(b) := \#\{R_\alpha \in C(n, m) \ : \ b \cap \Delta(R_\alpha, c \cdot h(\alpha)) \neq \emptyset\}, \tag{28}$$

and let

$$\tilde{q}_{n,m} := \sup\{\tilde{q}_{n,m}(b) \ : \ b \in \mathcal{B}(X), \mathrm{rad}(b) = \mathrm{rad}(B)R^{m-n}\}.$$

We have proved the following corollary.

Corollary 6.3. *Fix $B \in \mathcal{B}(X)$ and let the parameters R and c satisfy (25). Also, assume that for $n \in \mathbb{N}$ we have classes $C(n)$ defined by (26), each associated with a collection of subclasses $C(n,m)$ for $1 \leqslant m \leqslant n$. If for all pairs m,n and for some $\epsilon_0 > 0$ a splitting structure $(X,\mathcal{S},U,f)$ satisfies $\tilde{q}_{n,m} \leqslant R^{m(1-\epsilon_0)}$ then $\mathbf{Bad}(\mathcal{R},h)$ is ϵ_0-Cantor-winning on B with respect to $(X,\mathcal{S},U,f)$.*

The values $\tilde{q}_{n,m}(b)$ are usually easier to compute than $q_{n,m}(b)$. However, in some cases $\tilde{q}_{n,m}$ may become much larger than $q_{n,m}$ and so Corollary 6.3 will not be applicable. In that case, we will have to appeal to Corollary 6.2. As an example, we will encounter this phenomenon when we consider the standard set $\mathbf{Bad}_N$ and the p-adic set $\mathbf{Bad}_p(N)$ in the following section. The conditions in both Corollaries 6.2 & 6.3 should be compared with the conditions required in Theorem KTV.

7 Applications

7.1 Classical badly approximable points

We start this section with the model example of the classical set $\mathbf{Bad}_N$ of N-dimensional badly approximable points and describe how one can show it is Cantor-winning. Recall that $\mathbf{Bad}_N$ is known to be winning for both Schmidt's game and the HAW game, but is known not to be absolute winning for $N > 1$ (see [11] for example). As previously mentioned the set $\mathbf{Bad}_N$ can be written in the form of a generalized bad set with

$$\mathcal{R} = \{\mathbf{p}/q \;:\; \mathbf{p} \in \mathbb{Z}^N, q \in \mathbb{N}, \gcd(p_1,\dots,p_N,q) = 1\}$$

and $h(\mathbf{p}/q) = q^{-1-1/N}$. Let B be the unit box $B = [0,1]^N$. Next, choose an arbitrarily large R and a real number c such that $c^{-1} > \sqrt[N]{N!}3R^2$ (since $h(\mathbf{p}/q) \leqslant 1$ the condition $\sup_\alpha(c\cdot h(\alpha)) \leqslant R^{-1}$ is satisfied). It follows that

$$C(n) := \{\mathbf{p}/q \in \mathcal{R} \;:\; cR^n \leqslant q^{1+1/N} < cR^{n+1}\}. \tag{29}$$

Now, fix a ball b of $\mathrm{rad}(b) = R^{-n+1}$. If for $R_\alpha \in C(n)$ the neighbourhood $\Delta(R_\alpha, c\cdot h(\alpha))$ intersects b, then we have

$$\mathbf{d}(\mathrm{cent}(b), R_\alpha) \leqslant \mathrm{rad}(b) + c\cdot h(\alpha) < 3/2R^{-n+1}.$$

In other words, every element $R_\alpha \in C(n)$ for which the neighbourhood $\Delta(R_\alpha, c\cdot h(\alpha))$ intersects b must lie inside the ball of diameter $3R^{-n+1}$ centred at $\mathrm{cent}(b)$.

We reprove the classical Simplex Lemma. Assume that there are at least $N+1$ points $R_{\alpha_1},\dots,R_{\alpha_{N+1}}$ with $R_{\alpha_i} \in C(n)$ such that their neighbourhoods $\Delta(R_{\alpha_i}, c\cdot h(\alpha_i))$ intersect b. We compute the volume of the simplex with vertices at points $R_{\alpha_1},\dots,R_{\alpha_{N+1}}$. On the one hand this volume must be less than $3^N R^{-N(n-1)}$ since all of the vertices lie in the box of side length $3R^{-n+1}$ centred at $\mathrm{cent}(b)$. On the other hand the volume is either zero or is bounded below by $(N!\cdot q_1q_2\cdots q_{N+1})^{-1}$ where $R_{\alpha_i} = \mathbf{p}_i/q_i$. By (29) we have

$$\frac{1}{N!}(q_1q_2\cdots q_{N+1})^{-1} \geqslant \frac{1}{N!}c^{-N}R^{-N(n+1)} > 3^N R^{-N(n-1)},$$

which is impossible. Therefore, the area of the simplex must be zero; in other words, all points $R_{\alpha_1},\dots,R_{\alpha_{N+1}}$ must lie on some $(N-1)$-dimensional affine hyperplane in $\mathbb{R}^N$. If there are less than $N+1$ points $R_\alpha \in C(n)$ with $\Delta(R_\alpha, c\cdot h(\alpha)) \cap b \neq \emptyset$ then we can easily find an affine hyperplane containing all of them.

The upshot is that for each b of radius R^{-n+1} there exists an affine hyperplane $\mathcal{H}_b$ which contains all the points $R_\alpha \in C(n)$ such that $\Delta(R_\alpha, c \cdot h(\alpha)) \cap b \neq \emptyset$. Thus, define $C(n,1) := C(n)$ and for $2 \leqslant m \leqslant n$, $C(n,m) := \emptyset$ and consider the set

$$\mathcal{E}_b := \{E \in \mathcal{B}(\mathbb{R}^k) \ : \ E \subset b, \ \mathrm{rad}(E) = R^{-n}, \ \exists R_\alpha \in C(n), \ E \cap \Delta(R_\alpha, c \cdot h(\alpha)) \neq \emptyset\}.$$

It follows that every $E \in \mathcal{E}_b$ must intersect the $c \cdot h(\alpha)$-neighbourhood of $\mathcal{H}_b$. By construction we have that $q_{n,m}(b)$ represents the maximal number of disjoint balls in $\mathcal{E}_b$. The definition of $C(n)$ yields that $c \cdot h(\alpha) \leqslant R^{-n}$, and so $q_{n,1}(b) \ll R^{N-1}$. Furthermore, we have $q_{n,1} \ll R^{N-1}$. Note that for $m \geqslant 2$, the value of $q_{n,m}$ is surely zero.

Consider an arbitrary splitting structure $(\mathbb{R}^N, \mathcal{S}, U, f)$. If $d = \dim A_\infty(B) > N-1$ then by Corollary 2.4 we have $f(R) = R^d$ and

$$q_{n,1} \ll f(R)^{\frac{N-1}{d}} = f(R)^{1-\frac{d-N+1}{d}}.$$

This verifies the conditions of Corollary 6.2 for $\epsilon_0 = \frac{d-N+1}{d}$ and therefore $\mathbf{Bad}_N$ is $\frac{d-N+1}{d}$-Cantor-winning for $(\mathbb{R}^N, \mathcal{S}, U, f)$. In particular for the canonical splitting structure of $\mathbb{R}^N$, $\mathbf{Bad}_N$ is $1/N$-Cantor winning. This straightforwardly implies the following proposition.

Proposition 7.1. *The set $\mathbf{Bad}_N$ has full Hausdorff dimension; i.e.,* $\dim \mathbf{Bad}_N = N$. *Moreover, if (S4) holds for $\mathbb{R}^N$ with some splitting structure satisfying* $\dim A_\infty(B) > N-1$, *then*

$$\dim(\mathbf{Bad}_N \cap A_\infty(B)) = \dim A_\infty(B).$$

This result is not new. For example, in the case of the canonical splitting structure (so $A_\infty(B) = B$ for every B) this is simply the classical Jarník theorem, whereas in the case that $A_\infty(B)$ has strictly positive codimension and supports a measure satisfying (24) it is implied by the work of Fishman [17] (see also [22, Theorem 8]). However, a construction of Cantor-winning sets for more complicated generalized bad sets provides the answers to some open problems as we shall now see.

7.2 p-adically badly approximable numbers

We first demonstrate how our broad framework allows us to prove new results in spaces different to $\mathbb{R}^N$. Firstly, we consider the set $\mathbf{Bad}_p(N)$ of p-adically badly approximable vectors.

Theorem 7.1. *The set $\mathbf{Bad}_p(N)$ is $\frac{d-N+1}{d}$-Cantor-winning on any given ball $B \in \mathcal{B}(\mathbb{Z}_p^N)$ for any non-trivial splitting structure of $\mathbb{Z}_p^N$ satisfying $d = \dim A_\infty(B) > N-1$, so long as (S4) holds. In particular, the set $\mathbf{Bad}_p(N)$ is $\frac{1}{N}$-Cantor-winning with respect to the canonical splitting structure of $\mathbb{Z}_p^N$ induced from $\mathbb{Q}_p^N$.*

In particular, for the canonical splitting structure of $\mathbb{Z}_p^N$ Theorem 7.1 shows that the set $\mathbf{Bad}_p(N)$ has maximal Hausdorff dimension N, reproducing the results of [2] and [22]. However, to the best of the authors' knowledge, no winning-type results for $\mathbf{Bad}_p(N)$ were previously known.

Proof. Note that the radius of any ball in $\mathbb{Q}_p^N$ is an integer power of p. Therefore without loss of generality we will assume that in the proof the parameter R is always an integer power of p. Recall that $\mathbb{Z}_p^N$ comes equipped with a normalized Haar measure m such that the measure of each ball b is $m(b) = (\mathrm{rad}(b))^N$.

As discussed earlier, one can readily verify that the set $\mathbf{Bad}_p(N)$ is a generalised badly approximable set with

$$\mathcal{R} = \{\mathbf{r}/q \in \mathbb{Z}_p^N \ : \ \mathbf{r} = (r_1, \ldots, r_N) \in \mathbb{Z}^N, \ q \in \mathbb{N}\}$$

and $h(\mathbf{r}/q) = (\max\{|r_1|, \ldots, |r_N|, |q|\})^{-\frac{N+1}{N}} \cdot |q|_p^{-1}$. For simplicity we provide the proof for the particular ball $B = \mathbb{Z}_p^N$, the proof for other balls follow the same arguments. We therefore assume from here on that $\operatorname{diam}(B) = 1$. We may also assume that q is always coprime with p. Indeed, assuming otherwise we either have that one of r_i's in the definition (6) of $\mathbf{Bad}_p(N)$ is coprime with p, or all of them satisfy $|r_i|_p < 1$. In the first case the right hand side of (6) equals one and condition (6) itself is automatically verified with $c = 1$. In the second case the condition readily follows from one for the parameters $(r_1/p, r_2/p, \ldots, r_N/p, q/p)$. Whence the formula for the height may be simplified as follows:

$$h(\mathbf{r}/q) = (\max\{|r_1|, \ldots, |r_N|, |q|\})^{-\frac{N+1}{N}}.$$

Choose an arbitrarily large R, which is a power of p, and a sufficiently small c to be specified later. It follows that

$$C(n) = \{\mathbf{r}/q \in \mathcal{R} \ : \ c^{-1}R^{-n-1} < h(\mathbf{r}/q) \leqslant c^{-1}R^{-n}\}.$$

Now, fix a ball b of $\operatorname{rad}(b) = R^{-n+1}$. If for some $\mathbf{r}/q \in C(n)$ the neighbourhood $\Delta(\mathbf{r}/q, \ c \cdot h(\mathbf{r}/q))$ intersects b, then since $c \cdot h(\mathbf{r}/q) < R^{-n+1}$ it follows from the ultra-metric inequality that this neighbourhood must in fact be contained in b. In other words, every element $\mathbf{r}/q \in C(n)$ for which $\Delta(\mathbf{r}/q, \ c \cdot h(\mathbf{r}/q))$ intersects b must lie inside b.

Assume as in §7.1 that there are at least $N+1$ points $\mathbf{r}^{(1)}/q^{(1)}, \ldots, \mathbf{r}^{(N+1)}/q^{(N+1)}$ not all lying on some $(N-1)$-dimensional affine subspace of $\mathbb{Z}_p^N$ and such that their neighbourhoods $\Delta(\mathbf{r}^{(i)}/q^{(i)}, \ c \cdot h(\mathbf{r}^{(i)}/q^{(i)}))$ all intersect b. These $N+1$ points therefore span a p-adic simplex

$$\mathcal{S} := \left\{ \frac{\mathbf{r}^{(1)}}{q^{(1)}} + x_2 \left(\frac{\mathbf{r}^{(2)}}{q^{(2)}} - \frac{\mathbf{r}^{(1)}}{q^{(1)}} \right) + \ldots + x_{N+1} \left(\frac{\mathbf{r}^{(N+1)}}{q^{(N+1)}} - \frac{\mathbf{r}^{(1)}}{q^{(1)}} \right) : (x_2, \ldots, x_{N+1}) \in \mathbb{Z}_p^N \right\}.$$

which is contained in b. One can check that $\mathcal{S} - \frac{\mathbf{r}^{(1)}}{q^{(1)}} = \mathcal{M} \cdot \mathbb{Z}_p^N$, where $\mathcal{M}$ is a matrix given by

$$\mathcal{M} := \begin{pmatrix} r_1^{(2)}/q^{(2)} - r_1^{(1)}/q^{(1)} & \cdots & r_N^{(2)}/q^{(2)} - r_N^{(1)}/q^{(1)} \\ \vdots & & \vdots \\ r_1^{(N+1)}/q^{(N+1)} - r_1^{(1)}/q^{(1)} & \cdots & r_N^{(N+1)}/q^{(N+1)} - r_N^{(1)}/q^{(1)} \end{pmatrix}. \qquad (30)$$

Therefore the Haar measure of $\mathcal{S}$ is equal to $m(\mathcal{S}) = |\det \mathcal{M}|_p \cdot m(\mathbb{Z}_p^N) = |\det \mathcal{M}|_p$. It is easy to check using the definitions of $C(n)$ and the height function h that the determinant $\det \mathcal{M}$ takes the form of a non-zero rational number M/Q with denominator $Q = \prod_{i=1}^{N+1} q^{(i)}$ and numerator M satisfying

$$|M| < \left(c^{\frac{N}{N+1}} R^{\frac{(n+1)N}{N+1}} \cdot \right)^{N+1} \#S_{N+1} \ = \ (N+1)! \cdot c^N \cdot R^{(n+1)N},$$

where S_{N+1} is the symmetric group on $N+1$ symbols. Indeed, since we are assuming $|q^{(i)}|_p = 1$ for $i = 1, \ldots, N+1$ it follows that $|Q|_p = 1$ and therefore

$$m(\mathcal{S}) \geqslant |M|_p \ \geqslant \ |M|^{-1} \ > \ \frac{1}{(N+1)! \cdot c^N \cdot R^{(n+1)N}}.$$

Taking $c \leqslant \sqrt[N]{((N+1)!)^{-1} \cdot c_1} \cdot R^{-2}$ we reach a contradiction since the of Haar measure the ball b, which contains the simplex $\mathcal{S}$, equals $R^{-(n-1)N}$. Moreover, it is easy to see that criterion (25) is satisfied for this choice so long as R is sufficiently large.

We deduce that all of the points $\mathbf{r}^{(1)}/q^{(1)}, \ldots, \mathbf{r}^{(N+1)}/q^{(N+1)}$ must lie on some $(N-1)$-dimensional affine hyperplane in $\mathbb{Z}_p^N$. If there are less than $N+1$ points $\mathbf{r}/q \in C(n)$ with $\Delta(\mathbf{r}/q,\, c \cdot h(\mathbf{r}/q)) \cap b \neq \emptyset$ then we can easily find an affine hyperplane containing all of them. Thus, setting $C(n,1) := C(n)$ and $C(n,m) := \emptyset$ for $2 \leqslant m \leqslant n$, it follows from geometric arguments analogous to those exhibited in §7.1 that $q_{n,1}(b) \ll R^{N-1}$. Furthermore, we have $q_{n,1} \ll R^{N-1}$ and for $m \geqslant 2$ that $q_{n,m} = 0$. As before, this is enough to show that for any non-trivial splitting structure on $\mathbb{Z}_p^N$ with $d = \dim A_\infty(B) > N-1$ the conditions of Corollary 6.2 are satisfied with $\epsilon_0 = \frac{d-N+1}{d}$. Therefore, the set $\mathbf{Bad}_p(N)$ is $\frac{d-N+1}{d}$-Cantor-winning on the ball B.

□

7.3 The mixed Littlewood conjecture and the behavior of the Lagrange constant for multiples of a fixed irrational number

Recall that the Lagrange constant $c(\alpha)$ of an irrational number α is defined as the quantity

$$c(\alpha) := \liminf_{q \to \infty} q \cdot ||q\alpha||.$$

Obviously, $c(\alpha) > 0$ if and only if $\alpha \in \mathbf{Bad}$. On the other hand, a classical theorem of Dirichlet in the theory of Diophantine approximation implies that $c(\alpha)$ cannot exceed 1. In recent years there has been a surge of interest in investigating the behaviour of the Lagrange constant of multiples of α; that is, the behaviour of the sequence of real numbers $c(n\alpha)$ for $n \in \mathbb{N}$.

By denoting $q' = qn$ one can easily observe that

$$n \cdot \liminf_{q \to \infty} q \cdot ||q\alpha|| \geqslant \liminf_{q \to \infty} q \cdot ||q(n\alpha)|| = 1/n \cdot \liminf_{q' \to \infty} q' \cdot ||q'\alpha|| \geqslant 1/n \cdot \liminf_{q \to \infty} q \cdot ||q\alpha||,$$

which in turn shows that for any positive integer n and any badly approximable α we always have

$$\frac{c(\alpha)}{n} \leqslant c(n\alpha) \leqslant nc(\alpha).$$

In [5] the authors posed the following problem.

Problem A. *Is it true that every badly approximable real number α satisfies*

$$\lim_{n \to \infty} c(n\alpha) = 0\,?$$

By replacing n with powers of a prime number p the answer to this problem is equivalent to the well known p-adic Littlewood conjecture. It is the belief of the first author that the answer to Problem A is negative, although at the moment this problem remains open. The strongest related result currently found in the literature is due to Einsiedler, Fishman & Shapira [16]. They answered positively a weaker version of Problem A:

Theorem EFS. *Every badly approximable real number α satisfies*

$$\inf_{n \geqslant 1} c(n\alpha) = 0.$$

Using the framework layed out in this paper we can show that there are a multitude of numbers $\alpha \in \mathbb{R}$ for which the sequence $c(n\alpha)$ either does not tend to zero or tends to zero as slow as you wish.

Theorem 7.2. *For any function $g : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ such that $\lim_{q\to\infty} g(q) = \infty$, the set of real numbers $\alpha \in [0,1]$ satisfying the inequality*

$$\limsup_{k\to\infty} g(k) \cdot c(k\alpha) > 0$$

is 1-Cantor-winning for any non-trivial splitting structure of $\mathbb{R}$ for which (S4) holds.

Remark 7.1. It was recently pointed out in [12] that this result answers the dimension one case of the second part Problem 4.4 of Bugeaud's paper [13].

Proof. For any function g and large parameter R we will provide the sequence $(k_i)_{i\in\mathbb{N}}$ of positive integers such that $g(k_i) \cdot c(k_i\alpha) > c$ for some positive constant c, possibly dependent on α. Then one can easily see that the set of interest

$$\{\alpha \in \mathbb{R} \ : \ \exists c > 0, \forall (i,p,q) \in \mathbb{N}\times\mathbb{Z}\times\mathbb{N},\ g(k_i)\cdot q \cdot |qk_i\alpha - p| > c\} \tag{31}$$

is indeed a generalized bad set with $\mathcal{R} = \{R_{i,p,q,} = p/k_iq \ : \ (i,p,q) \in \mathbb{N}\times\mathbb{Z}\times\mathbb{N}\}$ and $h(i,p,q) = (g(k_i)k_iq^2)^{-1}$. The authors do not see a possibility to apply Theorem KTV for this set $\mathbf{Bad}(\mathcal{R},h)$, however we will show that Corollary 6.3 is applicable.

Consider the ball $B = [0,1]$, choose an arbitrary large parameter R and take $c = R^{-2}$. Then choose the values k_i such that $g(k_i) \geqslant R^{i-1}$ for every $i \in \mathbb{N}$. We can surely do this since $g(k) \to \infty$ as $k \to \infty$. Then let

$$C(n) = \left\{ \frac{p}{k_iq} \in \mathcal{R} \ : \ R^{n-2} \leqslant g(k_i)k_iq^2 < R^{n-1} \right\}.$$

We split the class $C(n)$ into subclasses in the following way. Set

$$C(n,m) := \{R_{i,p,q} \in C(n) \ : \ i = m\}.$$

Then, for any two different values $p_1/k_mq_1, p_2/k_mq_2$ from the same subclass $C(n,m)$ we have

$$\left| \frac{p_1}{k_mq_1} - \frac{p_2}{k_mq_2} \right| \geqslant \frac{1}{k_mq_1q_2} > \frac{g(k_m)}{R^{n-1}} \geqslant R^{-n+m}.$$

The final inequality automatically implies that for any ball b of radius R^{-n+m} we have

$$\#\{R_\alpha \in C(n,m) \ : \ b \cap \Delta(R_\alpha, c\cdot h(\alpha)) \neq \emptyset\} \ll 1;$$

or, in other words, in view of (28) we have $\tilde{q}_{n,m} \ll 1$. Thus, for any non-trivial splitting structure $(X,\mathcal{S},U,f)$ we have $\tilde{q}_{n,m} \ll f(R)^{1-1}$, the conditions of Corollary 6.3 are fulfilled and so the set $\mathbf{Bad}(\mathcal{R},h)$ exhibited in (31) is 1-Cantor-winning. □

The proof of Theorem 7.2 suggests that its statement remains valid even if we impose more restrictive conditions on α.

Theorem 7.3. *Let $g : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ be a function such that $\lim_{q\to\infty} g(q) = \infty$ and let $(k_i)_{i\in\mathbb{N}}$ be a sequence such that*

$$\lim_{i\to\infty} \frac{g(k_{i+1})}{g(k_i)} = \infty.$$

Then the set of $\alpha \in [0,1]$ such that

$$\inf_{i\in\mathbb{N}} g(k_i)c(k_i\alpha) > 0 \tag{32}$$

is 1-Cantor-winning for any non-trivial splitting structure of $\mathbb{R}$ for which (S4) holds.

Proof. Denote by W the set of α satisfying condition (32) and fix an arbitrary R. Then, there exists a value $i_0 = i_0(R)$ such that for every $i \geqslant i_0$ one has

$$\frac{g(k_{i+1})}{g(k_i)} \geqslant R.$$

Next, as in the previous proof we take $B = [0,1]$, $c = R^{-2}$ and $k'_i := k_{i_0+i}$ ($i \in \mathbb{N}$), so the condition $g(k'_i) \geqslant R^{i-1}$ is satisfied. Next, we split $\mathcal{R}$ into classes $C(n)$ and then into $C(n,m)$ as in the previous proof; that is, let

$$C(n) = \left\{ \frac{p}{k_i q} \in \mathcal{R} \ : \ R^{n-2} \leqslant g(k_i) k_i q^2 < R^{n-1} \right\}$$

and

$$C(n,m) := \{R_{i,p,q} \in C(n) \ : \ i = m\}.$$

Finally, by following the same arguments as in Theorem 7.2 we deduce that $\tilde{q}_{n,m} \ll 1$, which in turn implies that for any non-trivial splitting structure $(X, \mathcal{S}, U, f)$ we have $\tilde{q}_{n,m} \ll f(R)^{1-1}$. Finally, consider the set

$$W_R := \{\alpha \in \mathbb{R} \ : \ \exists c > 0, \forall i \in \mathbb{N}, \, g(k'_i) \cdot c(k'_i \alpha) > c\}$$

and notice that for any $\alpha \in W_R$ we have

$$\inf_{i \in \mathbb{N}} g(k_i) c(k_i \alpha) = \min_{1 \leqslant i \leqslant i_0} \{g(k_i) c(k_i \alpha), \inf_{j > i_0} \{g(k_j) c(k_j \alpha)\}\} = \min_{1 \leqslant i \leqslant i_0} \{g(k_i) c(k_i \alpha), c\} > 0.$$

Thus, for all sufficiently large R each set W_R is contained in W and this shows that W is indeed 1-Cantor-winning. □

An important application of Theorem 7.3 is that it can be applied to certain sets related to the Mixed Littlewood Conjecture introduced in Section 1.1. For a given function $g : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ and a sequence $\mathcal{D} = (d_n)_{n \geqslant 0}$ we define the set

$$\mathbf{Mad}_{\mathcal{D}}(g) := \{x \in \mathbb{R} \ : \ \liminf_{q \to \infty} q \cdot g(q) \cdot |q|_{\mathcal{D}} \cdot ||qx|| > 0\}.$$

The Mixed Littlewood Conjecture is then precisely the statement that $\mathbf{Mad}_{\mathcal{D}}(g)$ is empty when $g \equiv 1$ for any sequence $\mathcal{D}$. Very recently [7], the following result was proven.

Theorem BV. *Let $\mathcal{D} = (2^{2^n})_{n \in \mathbb{N}}$. Then, the set $\mathbf{Mad}_{\mathcal{D}}(g)$ has full Hausdorff dimension for $g(q) = \log\log q \cdot \log\log\log q$.*

With help of Theorem 7.3 we show that for a suitably chosen sequence $\mathcal{D}$ one may take an even slower growing function $g(q)$ than $\log\log q \cdot \log\log\log q$. In fact, one may choose a sequence $\mathcal{D}$ and a function $g(q)$ in such a way that g grows arbitrarily slowly yet $\mathbf{Mad}_{\mathcal{D}}(g)$ is still of full Hausdorff dimension. Recall that for a sequence $\mathcal{D} = (d_i)_{i \in \mathbb{N}}$ we define the quantities

$$D_0 := 1; \quad D_n := \prod_{k=1}^{n} d_k.$$

Corollary 7.4 (to Theorem 7.3)**.** *Let $g : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ be a function which monotonically tends to infinity. Then for every sequence $\mathcal{D} = (d_i)_{i \in \mathbb{N}}$ such that*

$$\lim_{i \to \infty} \frac{g(D_{i+1})}{g(D_i)} = \infty$$

the set $\mathbf{Mad}_{\mathcal{D}}(g)$ is 1-Cantor-winning.

Unfortunately, the condition $g(q) \to \infty$ is crucial for the proof and so this corollary does not provide any counterexample to mixed Littlewood conjecture itself. That said, the first author does believe that the conjecture is indeed false for sufficiently rapidly growing sequences $\mathcal{D}$.

Proof. For a given function g we take the sequence $(k_i)_{i\in\mathbb{N}} = \{D_i\}_{i\in\mathbb{N}}$ and consider the set W as in the proof of Theorem 7.3. It follows that the set W is 1-Cantor-winning. Finally, it suffices to check that $\mathbf{Mad}_{\mathcal{D}}(g)$ contains W. Indeed, consider $\alpha \in W$ and an arbitrary number q, and let $|q|_{\mathcal{D}} = k_i^{-1}$. This implies that $q = k_i q'$ and by the definition of the pseudo-norm it immediately follows that $q \geqslant k_i$. Therefore,

$$g(q) \cdot q \cdot |q|_{\mathcal{D}} \cdot ||q\alpha|| \geqslant g(k_i) \cdot q' \cdot ||q' \cdot (k_i\alpha)||.$$

The proof is complete upon application of condition (32). □

7.4 Multiplicative semigroups of integers (the $\times a$ $\times b$ problem).

In his remarkable work [19], Furstenberg showed that if a and b are multiplicatively independent positive integer numbers then for every irrational α the set

$$\{a^n b^m \alpha \pmod 1 \ : \ n, m \in \mathbb{N}\}$$

is dense in the unit interval. Later, Bourgain, Lindenstrauss, Michel & Venkatesh [10, Theorem 1.8] achieved a quantitative version of this result, which we formulate in the following way.

Theorem BLMV. *Let $\Sigma := \{a^n b^m : n, m \in \mathbb{Z}_{\geqslant 0}\}$ be a multiplicative semigroup generated by two multiplicatively independent integers a and b. Suppose that for irrational α there exists $k > 0$ so that*

$$\left|\alpha - \frac{p}{q}\right| \geqslant q^{-k}, \text{ for all } q \geqslant 2, p, q \in \mathbb{Z}.$$

Then there exists a positive constant $c = c(a, b)$ such that the inequality

$$||q\alpha|| < (\log\log\log q)^{-c}$$

is satisfied for infinitely many $q \in \Sigma$.

We will show that there are numbers for which $||q\alpha||$ can not be made too small. To be precise, given a function $g : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ we define

$$\mathbf{Bad}_{\times a, \times b}(g) := \{\alpha \in \mathbb{R} \ : \ \exists c > 0 \text{ s.t. } \forall q \in \Sigma, \ ||q\alpha|| \geqslant c \cdot (g(q))^{-1}\}.$$

Before stating the theorem we define a modified logarithm function. Let

$$\log^* q := \begin{cases} 1, & \text{if } q < e. \\ \log q, & \text{otherwise.} \end{cases}$$

Theorem 7.5. *For any pair a, b of multiplicatively independent positive integers the set $\mathbf{Bad}_{\times a, \times b}(g) \cap [0, 1]$ is $\frac{\epsilon}{1+\epsilon}$-Cantor-winning for $g(q) = (\log^* q)^{1+\epsilon}$, where ϵ is an arbitrary positive constant.*

Remark 7.2. By using similar methods to those used in [7] one can show that for $g_1(q) = \log^* q \cdot \log^* \log q$ the set $\mathbf{Bad}_{\times a, \times b}(g_1) \cap [0, 1]$ has in fact full Hausdorff dimension. However, this would not give us the Cantor winning property for $\mathbf{Bad}_{\times a, \times b}(g_1)$.

Proof. As before, we first represent the set $\mathbf{Bad}_{\times a,\times b}(g)$ as a generalized bad set. For this reason let

$$\mathcal{R} = \left\{ \frac{p}{q} \ ; \ p \in \mathbb{N}, q \in \Sigma \right\}$$

and $h(p/q) = (q \cdot g(q))^{-1}$. Consider the values

$$r(q) := \frac{g(q)}{g(q \cdot g(q))}$$

for $q \in \Sigma$. Obviously one has $r(q) < 1$, but on the other hand $g(q) < q$ for all $q > q_0(\epsilon)$. For these $q > q_0(\epsilon)$ we have

$$\frac{g(q)}{g(q \cdot g(q))} > \frac{g(q)}{g(q^2)} = \frac{g(q)}{2^{1+\epsilon} g(q)} = \frac{1}{2^{1+\epsilon}}.$$

Therefore, $r(q)$ is bounded from below by a positive constant which depends only on ϵ. Define constants $c_1 = c_1(\epsilon)$ and $c_2 = c_2(\epsilon)$ such that

$$c_1 := \min_{q \in \mathbb{N}} \{1/r(q)\}; \quad c_2 = \max_{q \in \mathbb{N}} \{1/r(q)\}.$$

For sufficiently large R the class $C(n)$ will take the form

$$C(n) := \{p/q \in \mathcal{R} \ : \ cR^n \leqslant q \cdot g(q) < cR^{n+1}\},$$

for some constant c to be specified later. It can be readily verified that $C(n)$ is contained within the possibly slightly larger class

$$C^*(n) := \left\{ p/q \in \mathcal{R} \ : \ \frac{c_1 \cdot cR^n}{g(cR^n)} \leqslant q < \frac{c_2 \cdot cR^{n+1}}{g(cR^{n+1})} \right\}. \tag{33}$$

Now we split $C^*(n)$ into subclasses $C^*(n,s)$ in the following way (note that these are not the subclasses $C(n,m)$ from the bad to Cantor set construction). Let

$$C^*(n,s) := \{p/(a^s b^t) \in C^*(n) \ : \ t \in \mathbb{Z}_{\geqslant 0}\}.$$

It is certainly the case that s is bounded below by zero. However, whenever $C^*(n,s) \neq \emptyset$ we have by (33) that

$$s \log a \leqslant \log \frac{c_2 c \cdot R^{n+1}}{g(cR^{n+1})}.$$

By choosing c small enough we can guarantee that $s \leqslant \frac{\log R}{\log a} \cdot n - 1$. This means that for a fixed n there are at most $\frac{\log R}{\log a} \cdot n$ various non-empty classes $C^*(n,s)$.

Next, consider two different elements $p_1/(a^s b^{t_1})$ and $p_2/(a^s b^{t_2})$ from $C^*(n,s)$. We have

$$\left| \frac{p_1}{a^s b^{t_1}} - \frac{p_2}{a^s b^{t_2}} \right| \geqslant \frac{1}{a^s b^{\max\{t_1,t_2\}}} \overset{(33)}{>} \frac{g(cR^{n+1})}{c_2 \cdot cR^{n+1}}.$$

By taking c small enough we can guarantee that the distance between two neighbouring numbers from $C^*(n,s)$ is at least $g(R^n)R^{-n+2\epsilon}$.

For convenience denote $k := R^{2\epsilon}$, and let m be the minimal positive integer satisfying

$$R^m \geqslant Rk \cdot g(R^n). \tag{34}$$

Then, it is surely the case that $R^m < R^2k \cdot g(R^n)$ and so

$$(n \log R)^\epsilon > \left(\frac{R^m}{R^2k}\right)^{\frac{\epsilon}{1+\epsilon}}. \tag{35}$$

There must exist a natural number $n_0(R)$ (or more exactly $n_0(R,\epsilon)$) such that for each $n \geqslant n_0(R)$ the value m is no larger than n. Again, by choosing c small enough we are able to guarantee that $C(n) = \emptyset$ for all $n < n_0(R)$ and therefore can assume here-on that $n \geqslant n_0(R)$.

Consider any ball b of radius R^{-n+m}. Then,

$$\#\{R_\alpha \in C^*(n,s) \ : \ b \cap \Delta(R_\alpha, c \cdot h(\alpha)) \neq \emptyset\} \leqslant \frac{\mathrm{diam}(b)}{kg(R^n)R^{-n}} + 2.$$

Since $\mathrm{diam}(b) = R^{-n+m}$, and by the choice of m, the first summand on the r.h.s. is at least R and therefore for $R \geqslant 2$ we have that the r.h.s is bounded above by

$$\frac{2\mathrm{diam}(b)}{kg(R^n)R^{-n}}.$$

Now, by collecting all classes $C^*(n,s)$ together we have

$$\begin{aligned}\#\{R_\alpha \in C(n) \ : \ b \cap \Delta(R_\alpha, c \cdot h(\alpha)) \neq \emptyset\} &\leqslant \frac{2R^m}{k(n \log R)^{1+\epsilon}} \cdot \frac{n \log R}{\log a} \\ &\overset{(35)}{<} \frac{2(R^2k)^{\frac{\epsilon}{1+\epsilon}}}{k \log a} R^{m(1-\frac{\epsilon}{1+\epsilon})} \ll R^{m(1-\frac{\epsilon}{1+\epsilon})}. \end{aligned} \tag{36}$$

Finally, we are ready to split $C(n)$ into subclasses to finish the proof. Define $C(n,m)$ to be the empty set for every $n < n_0(R)$ and for every $m \neq m_0$ for m_0 given by (34). Let $C(n, m_0) = C(n)$. Then, inequality (36) implies that $\tilde{q}_{n,m} \ll R^{m(1-\epsilon/(1+\epsilon))}$ and application of Corollary 6.3 yields that the set $\mathbf{Bad}_{\times a \times b}(g)$ is $\frac{\epsilon}{1+\epsilon}$-Cantor-winning. □

7.5 Further examples

In several recent papers constructions similar to generalized Cantor sets were made inside other sets falling into the class of generalized bad sets. With a bit of effort one can prove a Cantor-winning property for the sets in question.

The set of points in $\mathbf{Bad}(i,j)$ lying on vertical lines.

Consider a pair (i,j) of non-negative real numbers such that $i + j = 1$. Let L_x be a vertical line passing through the point $(x, 0)$, where x satisfies the condition

$$\liminf_{q \to \infty} q^{1/i} \cdot ||qx|| > 0. \tag{37}$$

To prove Schmidt's conjecture in [6] the authors essentially applied a generalized Cantor set construction. To be exact, Theorem 4 and statement (26) from [6] immediately imply the following.

Proposition BPV. *The projection of* $\mathbf{Bad}(i,j) \cap \mathrm{L}_x$ *onto the y-axis is* ϵ_0*-Cantor-winning for* $\epsilon_0 = \frac{1}{32}(ij)^2$.

Once Proposition BPV is established one can immediately prove a result concerning the non-empty intersection of sets $\mathbf{Bad}(i,j)$ for various pairs (i,j). This was essentially the statement of Schmidt's conjecture.

Theorem BPV. *Let $((i_\alpha, j_\alpha))_{\alpha\in S}$ be a sequence of pairs of positive real numbers indexed by a finite or countable set S such that $i_\alpha + j_\alpha = 1$. Define*

$$\epsilon_0 := \inf\left\{\frac{1}{32}(i_\alpha j_\alpha)^2 \ : \ \alpha \in S\right\}.$$

Assume that $\epsilon_0 > 0$. Then for every $x \in \mathbb{R}$ satisfying (37), the projection of

$$\bigcap_{\alpha\in S} \mathbf{Bad}(i_\alpha, j_\alpha) \cap \mathrm{L}_x$$

onto the y axis is ϵ_0-Cantor-winning.

Sets $\mathbf{Bad}(i_1, i_2, \ldots, i_N)$ on non-degenerate curves

Later, in [8], the authors demonstrated that a result similar to Theorem BPV holds for the sets $\mathbf{Bad}(i, j) \cap \mathcal{C}$ for any non-degenerate planar curve. Independently [9] Beresnevich proved more general result in higher dimensions:

Let a curve $\mathcal{C}$ be parameterized by a map

$$\mathbf{f} \ : \ I \to \mathbb{R}^N; \quad \mathbf{f} \in C^N(I),$$

where $I \subset \mathbb{R}$ is some interval. We assume that $\mathbf{f}$ is non-degenerate at every point on I or, equivalently, that the Wronskian of $f'_1, \ldots, f'_N$ is not zero at every point $x \in I$. Let $i_1, i_2, \ldots, i_N$ be positive real numbers such that $i_1 + \ldots + i_N = 1$. Proposition 3 from [9] implies the following.

Theorem B. *The set*

$$\{x \in I \ : \ \mathbf{f}(x) \in \mathbf{Bad}(i_1, \ldots, i_N)\}$$

is ϵ_0-Cantor-winning where

$$\epsilon_0 = \min\left\{(2N)^{-4}, \frac{1 - (1 + \min\{i_k \ : \ 1 \leqslant k \leqslant N\})^{-1}}{2}\right\}.$$

Theorem B immediately gives a positive answer to a problem raised by Davenport: that there are uncountably many points from $\mathbf{Bad}(i_1, \ldots, i_N)$ on any non-degenerate curve. In fact, the set of such points has full Hausdorff dimension. Moreover with some effort (see the section of [9] entitled 'Theorem 2 implies Theorem 1') Theorem B implies that the dimension of points from $\mathbf{Bad}(i_1, \ldots, i_N)$ on any non-degenerate manifold $\mathcal{M}$ is of full Hausdorff dimension; i.e.

$$\dim(\mathbf{Bad}(i_1, \ldots, i_N) \cap \mathcal{M}) = \dim \mathcal{M}.$$

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