A Pricing Formula for Delayed Claims

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Abstract

We consider the valuation of contingent claims with delayed dynamics in a Black & Scholes complete market model. We find a pricing formula that can be decomposed into terms reflecting the market values of the past and the present, showing how the valuation of future cashflows cannot abstract away from the contribution of the past. As a practical application, we provide an explicit expression for the market value of human capital in a setting with wage rigidity.

Keywords— Stochastic functional differential equations, delay equations, no-arbitrage pricing, human capital, sticky wages.

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1 Introduction

It is a standard result in asset pricing theory that the absence of arbitrage opportunities is essentially equivalent to the existence of an equivalent probability measure under which the price of any contingent claim is a local martingale after deflation

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by the money market account; see [14, 20, 21]. In this paper we preserve the standard formulation of arbitrage pricing in a complete market model with security prices evolving as geometric Brownian motions (GBM). The main novelty of our work is that we consider contingent claims that have dynamics described by a stochastic functional differential equation (SFDE).

It is perhaps surprising that using the no-arbitrage pricing machinery we are able to derive an explicit valuation formula in the case of dynamics with memory, which is notoriously difficult to study. In particular, we show that the price can be decomposed into a term related to the 'current market value of the past' (in a sense to be made precise below), and a term reflecting the 'market value of the present'. In our setting the contribution of the past is represented by the portion of a contingent claim's past trajectory that shapes its dynamics going forward.[1] Using our pricing formula, we demonstrate that in the market consistent valuation of future cashflows the contribution of the past cannot be neglected.

As a practical application of our results, we consider in detail the case in which the contingent claim represents stochastic wages received by an agent over his/her lifetime (e.g., [17, 6]). It is well known that when labor income is spanned by tradable assets, the market value of human capital can be easily derived via risk-neutral valuation. In [17] this result is extended to take into account endogenous retirement and borrowing constraints. It is in general difficult to allow for richer dynamics of labor income, including unspanned sources of risk (e.g., [33]), or state variables capturing wage rigidity (e.g., [17], section 6). The empirical literature on labor income dynamics widely relies on auto-regressive moving average (ARMA) processes (e.g., [28], [1], [22], [31]): Reiss [35], Lorenz [26], and Dunsmuir *et al.* [16] show how SFDEs can be understood as the weak limit of discrete time ARMA processes. We therefore consider the introduction of delayed drift and volatility coefficients in a GBM labor income model to provide a tractable example of wage dynamics that adjusts slowly to financial market shocks. We obtain a closed form solution for human capital, which makes explicit the contributions of the market value of the past and the present. Our results demonstrate that SFDEs are valuable modelling tools that can address the findings of a large body of empirical literature on wage rigidity (e.g., [30], [25], [13], [3], [27]).

Although we discuss the human capital application extensively, the extension to other applications is immediate. For instance, we provide some references to the

[1] The importance of the past in understanding the qualitative feature of a model with delay was also emphasized in Fabbri and Gozzi [18], although in a deterministic setting, when solving the endogenous growth model with vintage capital of Boucekkine *et al.* [7].

literature on counterparty risk and derivatives valuation, in which analogous dynamics arise in the context of collateralization procedures entailing a delay in the marking-to-market procedure of over-the-counter derivatives (e.g., [9, 10]).

It should be noted that no-arbitrage pricing in the case of delayed price dynamics has been recently studied by many authors, see for example [2, 29]. Their focus however is on proving completeness of the market, hence very different from ours. On the other hand, their work suggests that our results are of broader applicability, in particular to settings where market completeness is preserved, such as the case in which tradable assets have delayed drift and volatility terms.

The paper is organized as follows. In the following section, we introduce the setup, and state our main result. Section 3 presents mathematical tools used to deal with the non-Markovian nature of a setting with delayed dynamics. In particular, we embed our problem in an infinite dimensional Hilbert space, on which the state variable process is Markovian. In section 4 we prove our results by following a chain of five lemmas. Section 5 concludes.

2 Setup and Main Result

Consider a Black-Scholes complete market model defined on our filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P})$. Available for trade are a money market account, S_0, and n risky assets with price vector process S. Prices have dynamics described by

$$\begin{cases} \mathrm{d}S_0(t) = & S_0(t) r \mathrm{d}t, \\ \mathrm{d}S(t) = & \mathrm{diag}(S(t)) \left\{ \mu \mathrm{d}t + \sigma \mathrm{d}Z(t) \right\}, \\ S_0(0) = & 1, \qquad S(0) \in \mathbf{R}^n_{>0}, \end{cases} \tag{1}$$

where Z is an n-dimensional Brownian motion, $\mu \in \mathbf{R}^n$, and $\sigma \in \mathbf{R}^n \otimes \mathbf{R}^n$, such that $\sigma\sigma^\top > 0$. Here and in what follows, we use the notation $\mathbf{R}^n_{>0}$ for the set $\{(x_i) \in \mathbf{R}^n : x_i > 0, \ i = 1, \dots, n\}$. We assume that $\mathbf{F} := (\mathcal{F}_t)_{t \geq 0}$ is the filtration generated by the Brownian Motion Z, and enlarged with the $\mathbf{P}$-null sets. Defining the market price of risk as

$$\kappa := (\sigma^\top)^{-1}(\mu - r\mathbf{1}), \tag{2}$$

the stochastic discount factor ξ can be shown to evolve as follows in our setting (see [15]):

$$\begin{cases} \mathrm{d}\xi(t) & = -\xi(t) r \mathrm{d}t - \xi(t) \kappa^\top \mathrm{d}Z(t) \\ \xi(0) & = 1. \end{cases} \tag{3}$$

We consider the valuation of a payment stream represented by the $\mathbf{F}$-adapted process X_0. Our aim is to give an explicit expression to the following expectation:

$$HC\left(t_0\right) := \xi(t_0)^{-1}\mathbf{E}\left(\int_{t_0}^{+\infty} \xi(t)X_0(t)\mathrm{d}t\Big|\mathcal{F}_{t_0}\right). \tag{4}$$

The payment stream can be thought of as capturing the mark-to-market process of a trading account, the flow of profits and losses from a trading strategy, the collateral flows arising from an over-the-counter derivative transaction, or the labor income received by an agent over time. In the latter case, expression (4) represents the market value of the agent's human capital (e.g., [17]), which could be extended to a bounded horizon to model permanent exit from the labor market (e.g., death, irreversible unemployment or retirement) along the lines indicated in Remark 2.3 below. We assume that the payment stream X_0 obeys the following stochastic functional differential equation (SFDE):

$$\begin{cases} \mathrm{d}X_0(t) = & \left[X_0(t)\mu_0 + \int_{-d}^0 X_0(t+s)\phi(\mathrm{d}s)\right]\mathrm{d}t \\ & + \left[X_0(t)\sigma_0^\top + \begin{pmatrix} \int_{-d}^0 X_0(t+s)\varphi_1(\mathrm{d}s) \\ \vdots \\ \int_{-d}^0 X_0(t+s)\varphi_n(\mathrm{d}s)\end{pmatrix}^\top\right]\mathrm{d}Z(t), \\ X_0(0) = & x_0, \\ X_0(s) = & x_1(s) \quad \text{for } s\in[-d,0), \end{cases} \tag{5}$$

where $\mu_0 \in \mathbf{R}$, $\sigma_0 \in \mathbf{R}^n$, ϕ, φ_i are signed measures of bounded variation on $[-d,0]$ with $i = 1,\ldots,n$, and $x_0 \in \mathbf{R}_{>0}$, $x_1 \in L^2\big([-d,0];\mathbf{R}_{>0}\big)$. Note that when the payment stream is understood as labor income, then the SFDE introduces slow adjustment of wages to market shocks via delay terms in the drift and volatility coefficients of a GBM model. This provides a tractable model to capture the empirical evidence on wage rigidity discussed in the introduction.

Equation (5) admits a unique strong solution, as ensured by Theorem I.1 and Remark 4 Section I.3 in [32], and provides a simple, tractable example of income dynamics adjusting slowly to financial market shocks. Under dynamics (5), the valuation of (4) can be carried out within a complete market model characterised by a unique stochastic discount process ξ. The results of [2] and [29] suggest that the same applies to the more general setting in which the risky assets dynamics feature drift and volatility terms with memory.

To provide an explicit expression for (4), and formulate the main result of this paper, we define the functions

$$K(\lambda) := \lambda - (\mu_0 - \sigma_0^\top \kappa) - \int_{-d}^{0} e^{\lambda\tau} \Phi(\mathrm{d}\tau) \,, \quad \lambda \in \mathbf{C} \,, \tag{6}$$

$$\tilde{K}(\lambda) := \lambda - (\mu_0 - \sigma_0^\top \kappa) - \int_{-d}^{0} e^{\lambda\tau} |\Phi|(\mathrm{d}\tau) \,, \quad \lambda \in \mathbf{C} \,, \tag{7}$$

where the measure Φ on $[-d, 0]$ is given by

$$\Phi(\cdot) := \left[\phi(\cdot) - \begin{pmatrix} \varphi_1(\cdot) \\ \vdots \\ \varphi_n(\cdot) \end{pmatrix}^\top \kappa \right] , \tag{8}$$

and by $|\Phi|$ we mean the total variation measure of Φ.
We also define the constants

$$K := K(r) = r - \mu_0 + \sigma_0^\top \kappa - \int_{-d}^{0} e^{r\tau} \Phi(\mathrm{d}\tau), \tag{9}$$

$$\tilde{K} := \tilde{K}(r) = r - \mu_0 + \sigma_0^\top \kappa - \int_{-d}^{0} e^{r\tau} |\Phi|(\mathrm{d}\tau). \tag{10}$$

and assume the following conditions to hold throughout the paper.

Hypothesis 1. *(i) Φ is a signed measure of bounded variation on $[-d, 0]$,*

(ii) $\tilde{K}$ is strictly positive, i.e.

$$\tilde{K} > 0 \,. \tag{11}$$

We are now ready to state our main result, which provides an explicit decomposition of the market value of contingent payment stream X_0 in our setting.

Theorem 2.1. *Let ξ be defined as in (3), and X_0 evolve as in (5). Then, under Hypothesis 1, for any $t_0 \geq 0$ we can write*

$$HC(t_0) = \frac{1}{K} \left(X_0(t_0) + \int_{-d}^{0} G(s) X_0(t_0 + s) \, ds \right) , \quad \mathbf{P} - a.s., \tag{12}$$

where $X_0(t)$ denotes the solution at time t of equation (5)*, K is defined in* (9)*, and G is given by*

$$G(s) := \int_{-d}^{s} e^{-r(s-\tau)}\Phi(d\tau). \tag{13}$$

In expression (12), we recognize an annuity factor, K^{-1}, multiplying a term representing current value of X_0, and a term representing the current market value of the past trajectory of X_0 over the time window $(t_0 - d, t_0)$. The 'market value of the past' trades off the returns on the payment stream against its exposure to financial risk, as can be seen from expression (8). When the delay terms in the drift and volatility coefficients vanish, the valuation of the payment stream reduces to $K^{-1}X_0(t_0)$. Whereas Hypothesis 1 is all we need to provide the explicit valuation result of Theorem 2.1, the particular application to human capital requires labor income to be positive almost surely. A sufficient condition for this to be the case is provided in the next remark.

Remark 2.2. *A sufficient condition for almost sure positivity of X_0 is that $\phi \geq 0$ and $\varphi_i = 0$ for all i, so that the delay term in the volatility coefficient of* (5) *vanishes, and hence Φ coincides with ϕ and is nonnegative. Defining*

$$\mathcal{E}(t) := e^{(\mu_0 - \frac{1}{2}\sigma_0^\top \sigma_0)t + \sigma_0 Z(t)},$$
$$\mathcal{I}(t) := \int_0^t \mathcal{E}^{-1}(u) \int_{-d}^{0} X_0(s+u)\phi(ds)\, du,$$

the variation of constants formula yields

$$X_0(t) = \mathcal{E}(t)\big(x_0 + \mathcal{I}(t)\big), \tag{14}$$

which shows the positivity of labor income X_0 in this special case, as we are considering strictly positive initial conditions $x_0 \in \mathbf{R}_{>0}$ and $x_1 \in L^2\big([-d,0];\mathbf{R}_{>0}\big)$.

Remark 2.3. *The setup can be extended to the case of payments over a bounded horizon in some interesting situations. When the payment stream is received until an exogenous Poisson stopping time τ (representing death or irreversible unemployment, for example, in the case of labor income), expression* (12) *still applies, provided discounting is carried out at rate $r + \delta$ instead of r, where $\delta > 0$ represents the Poisson parameter.*

Example 2.4. *As a simple example of application of our setup to the context of over-the-counter derivatives, in equation* (5) *consider the case of $n = 1$, $\mu_0 = 0$, $\phi = 0$,*

$\sigma_0 = 0$, and $\varphi(s) = \delta_{-d}(s)$, where $\delta_a(s)$ indicates the delta-Dirac measure at a, so that equation (5) *reads*

$$dX_0(t) = X_0(t-d)\,dZ(t). \tag{15}$$

Then, for $t \in [0, d)$ we have

$$X_0(t) = x_0 + \int_0^t X_0(s-d)\,dZ(s) = x_0 + \int_{-d}^{t-d} x_1(\tau)\,dZ(\tau + d). \tag{16}$$

In this case $X_0(t)$ is Gaussian, and dynamics (15) *could be used to model, for example, the variation margin of an over-the-counter swap, when the collateralization procedure relies on a delayed mark-to-market value of the instrument (see [9], page 316, or [10], for example).*

3 Mathematical tools

It will be convenient to embed the labor income X_0 in the infinite dimensional Hilbert space $\mathcal{H}$

$$\mathcal{H} := \mathbf{R} \times L^2\big([-d, 0]; \mathbf{R}\big),$$

endowed with an inner product for $x = (x_0, x_1), y = (y_0, y_1) \in \mathcal{H}$ defined as

$$\langle x, y\rangle_{\mathcal{H}} := x_0 y_0 + \langle x_1, y_1\rangle_{L^2},$$

where

$$\langle x_1, y_1\rangle_{L^2} := \int_{-d}^{0} x_1(s) y_1(s)\,\mathrm{d}s.$$

In what follows we omit the subscript L^2 in the inner product notation.
Let us define two operators, A and C, that act on the domain $\mathcal{D}(A)$ as follows:[2]

$$\mathcal{D}(A) = \mathcal{D}(C) := \{(x_0, x_1) \in \mathcal{H} : x_1 \in W^{1,2}\big([-d, 0];\, \mathbf{R}\big), x_0 = x_1(0)\},$$

[2]The Sobolev space $W^{1,2}\big([-d, 0]; \mathbf{R}\big)$ is defined as

$$W^{1,2}\big([-d, 0]; \mathbf{R}\big) := \big\{u \in L^2([-d, 0]) :\ \exists\, g \in L^2([-d, 0]) \text{ such that } u(\theta) = c + \int_{-d}^{\theta} g(s)\,\mathrm{d}s\big\}.$$

and

$$A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H},$$
$$A(x_0, x_1) := \big(\mu_0 x_0 + \int_{-d}^{0} x_1(s)\phi(\mathrm{d}s), \frac{\mathrm{d}x_1}{\mathrm{d}s}\big),$$

with μ_0 and ϕ as in (5), and

$$C : \mathcal{D}(A) \subset \mathcal{H} \to \mathbf{R}^n \times L^2([-d, 0]; \mathbf{R}^n),$$
$$C\,(x_0, x_1) := \left(\sigma_0 x_0 + \begin{pmatrix} \int_{-d}^{0} x_1(s)\varphi_1(\mathrm{d}s) \\ \vdots \\ \int_{-d}^{0} x_1(s)\varphi_n(\mathrm{d}s) \end{pmatrix}, 0 \right),$$

with σ_0 and φ_i as in (5). The following, well known fact (see [12]) is crucial for the rest of the paper.

Lemma 3.1. *The operator A generates a strongly continuous semigroup in $\mathcal{H}$.*

Proof. The operator A can be written in the form

$$A\,(x_0, x_1) = \left(\int_{-d}^{0} x_1(\theta)a(\mathrm{d}\theta), \frac{\mathrm{d}x_1}{\mathrm{d}s} \right), \tag{17}$$

where

$$a(\mathrm{d}\theta) = \mu_0 \delta_0(\mathrm{d}\theta) + \phi(\mathrm{d}\theta)\,,$$

and δ_0 is the delta-Dirac measure at zero. The measure a defines a finite measure on $[-d, 0]$ and the lemma follows immediately from Proposition A.25 in [12]. □

The labor income in (5) can be equivalently defined as the first component of the solution in $\mathcal{H}$ of the following equation (see [11])

$$\begin{cases} \mathrm{d}X(t) & = AX(t)\mathrm{d}t + (CX(t))^\top \mathrm{d}Z(t), \\ X_0(0) & = x_0, \\ X_1(0, s) & = x_1(s) \text{ for } s \in [-d, 0), \end{cases} \tag{18}$$

with A and C defined above, and x_0, x_1 as in (5).

4 Proof of the Main Result

The proof of Theorem 2.1 will follow by a chain of five lemmas stated below. To prove the theorem we will consider the conditional mean of the labor income X_0 under an equivalent probability measure. We will show that this quantity obeys a deterministic differential equation described in terms of the operator A_1 defined below. Let

$$\mathcal{D}(A_1) := \left\{(x_0, x_1) \in \mathcal{H} : x_1(\cdot) \in W^{1,2}\big([-d,0];\mathbf{R}\big),\ x_0 = x_1(0)\right\},$$

and

$$\begin{aligned} &A_1 : \mathcal{D}(A_1) \subset \mathcal{H} \longrightarrow \mathcal{H} \\ &A_1(x_0, x_1) := \left((\mu_0 - \sigma_0^\top \kappa)x_0 + \int_{-d}^{0} x_1(s)\Phi(\mathrm{d}s), \frac{\mathrm{d}x_1}{\mathrm{d}s}\right), \end{aligned} \tag{19}$$

with $(\mu_0 - \sigma_0^\top \kappa) \in \mathbf{R}$ and Φ defined in (8). Replacing μ_0 with $\mu_0 - \sigma_0^\top \kappa$ and ϕ with Φ we infer from Lemma 3.1 and Hypothesis 1*(ii)* that A_1 generates a strongly continuous semigroup $(S(t))$ in $\mathcal{H}$. Let $\big(M_0(t;0,m_0,m_1),\ M_1(t,s;0,m_0,m_1)\big)$ be the solution at time t of the following differential equation

$$\begin{cases} \dfrac{\mathrm{d}M(t)}{\mathrm{d}t} = A_1 M(t), \\ \\ M_0(0) = m_0, \\ \\ M_1(0,s) = m_1(s), \quad s \in [-d,0), \end{cases} \tag{20}$$

with $m_0 \in \mathbf{R}_{>0}$ and $m_1 \in L^2\big([-d,0];\mathbf{R}_{>0}\big)$. Then by definition

$$S(t)\begin{pmatrix} m_0 \\ m_1 \end{pmatrix} = \begin{pmatrix} M_0(t;0,m_0,m_1) \\ M_1(t,s;0,m_0,m_1) \end{pmatrix}. \tag{21}$$

Denote by $\rho(A_1)$ and $R(\lambda, A_1) = (\lambda - A_1)^{-1}$, the resolvent set and the resolvent of A_1 respectively and by $\sigma(A_1)$ the spectrum of A_1. It is known (see for example Proposition 2.13 on p. 126 of [4] or Proposition A.25 in [12]) that the spectrum of A_1 is given by

$$\sigma(A_1) = \{\lambda \in \mathbf{C} :\ K(\lambda) = 0\},$$

where $K(\cdot)$ is defined in (6). Moreover it is known that $\sigma(A_1)$ is a countable set and every $\lambda \in \sigma(A_1)$ is an isolated eigenvalue of finite multiplicity. Let

$$\lambda_0 = \sup\{\mathrm{Re}\,\lambda :\ K(\lambda) = 0\} \tag{22}$$

be the spectral bound of A_1.

At this point, in order to prove the chain of lemmas (that we employ to prove Theorem 2.1) we need to introduce a new operator $\tilde{A}_1$. Let

$$\mathcal{D}(\tilde{A}_1) = \mathcal{D}(A_1),$$

and

$$\tilde{A}_1 : \mathcal{D}(A_1) \subset \mathcal{H} \to \mathcal{H},$$
$$\tilde{A}(x_0, x_1) := \big((\mu_0 - \sigma_0^T \kappa)x_0 + \int_{-d}^{0} x_1(s)|\Phi|(\mathrm{d}s), \frac{\mathrm{d}x_1}{\mathrm{d}s}\big),$$

Appealing to Lemma 3.1 we infer that $\tilde{A}_1$ generates a strongly continuous semigroup in $\mathcal{H}$. Denote by $\rho(\tilde{A}_1)$ and $\tilde{R}(\lambda, \tilde{A}_1) = (\lambda - \tilde{A}_1)^{-1}$, the resolvent set and the resolvent of $\tilde{A}_1$ respectively and by $\sigma\left(\tilde{A}_1\right)$ the spectrum of $\tilde{A}_1$. Arguing as for A_1 we have that the spectrum of $\tilde{A}_1$ is given by

$$\sigma\left(\tilde{A}_1\right) = \left\{\lambda \in \mathbf{C} : \ \tilde{K}(\lambda) = 0\right\},$$

where $\tilde{K}(\cdot)$ is defined in (7). $\sigma\left(\tilde{A}_1\right)$ is a countable set and every $\lambda \in \sigma\left(\tilde{A}_1\right)$ is an isolated eigenvalue of finite multiplicity. Let

$$\lambda_1 = \sup\left\{\operatorname{Re}\lambda : \ \tilde{K}(\lambda) = 0\right\} \tag{23}$$

be the spectral bound of $\tilde{A}_1$.

Lemma 4.1. *The function*

$$\mathbf{R} \ni \xi \longrightarrow \tilde{K}(\xi) \in \mathbf{R},$$

is strictly increasing and the spectral bound λ_1 is the only real root of the equation $\tilde{K}(\xi) = 0$. In particular, $\tilde{K}$ defined by (10) *is positive if and only if $r > \lambda_1$.*

Proof. The function $\tilde{K}(\cdot) : \mathbf{R} \to \mathbf{R}$ is differentiable and

$$\tilde{K}'(\xi) = 1 + \int_{-d}^{0} e^{\xi\tau}|\tau|\,|\Phi|(\mathrm{d}\tau) > 0, \quad \xi \in \mathbf{R}.$$

It is easy to see that

$$\lim_{\xi \to \pm\infty} \tilde{K}(\xi) = \pm\infty,$$

and therefore the equation $\tilde{K}(\xi) = 0$ has exactly one real solution ξ_0. Clearly, we have $\xi_0 \le \lambda_1$. To show that $\xi_0 = \lambda_1$ consider an arbitrary $\lambda = x + iy$ such that $\tilde{K}(\lambda) = 0$. Then

$$\begin{aligned} 0 &= x - \mu_0 + \sigma_0^\top \kappa - \int_{-d}^{0} e^{x\tau} \cos(y\tau)\, |\Phi|(\mathrm{d}\tau) \\ &\ge x - \mu_0 + \sigma_0^\top \kappa - \int_{-d}^{0} e^{x\tau}\, |\Phi|(\mathrm{d}\tau) \\ &= \tilde{K}(x)\,. \end{aligned}$$

Therefore, $\tilde{K}(x) \le 0$ which yields $x = \operatorname{Re}\lambda \le \xi_0$, hence $\lambda_1 \le \xi_0$. Finally, exploiting the fact that $\tilde{K}$ is an increasing function, we immediately get $\lambda_1 < r$ if and only if $\tilde{K}(r) > 0$. $\square$

Lemma 4.2. *Let K and $\tilde{K}$ be defined as in* (6) *and* (7) *and let λ_0 and λ_1 be the spectral bounds of the operators A_1 and $\tilde{A}_1$ (respectively). It holds*

$$\lambda_1 \ge \lambda_0.$$

Proof. Exploiting the fact that $\tilde{K}$ is an increasing function (see Lemma 4.1), in order to prove that $\lambda_0 \le \lambda_1$, it is sufficient to prove $\tilde{K}(\lambda_0) \le \tilde{K}(\lambda_1)$. Recall that from Lemma 4.1 we have that $\lambda_1 \in \mathbb{R}$ and actually λ_1 coincides with the only real root of the equation $\tilde{K}(\lambda) = 0$. Therefore, we just have to prove that $\tilde{K}(\lambda_0) \le 0$.
Let $\lambda = x + iy$ be a complex root of $K(\lambda) = 0$. In particular this means that its real part satisfies the following equation

$$x - (\mu_0 + \sigma_0^T \kappa) - \int_{-d}^{0} e^{x\tau} \cos(y\tau)\, \Phi(\mathrm{d}\tau) = 0.$$

Let us show that $\tilde{K}(\operatorname{Re}(\lambda)) = \tilde{K}(x) \le 0$. Keeping in mind the previous equality, we have that

$$\begin{aligned} \tilde{K}(x) &= x - (\mu_0 + \sigma_0^T \kappa) - \int_{-d}^{0} e^{x\tau} |\Phi|(\mathrm{d}\tau) \\ &= x - (\mu_0 + \sigma_0^T \kappa) - \int_{-d}^{0} e^{x\tau} \cos(y\tau)\, \Phi(\mathrm{d}\tau) - \int_{-d}^{0} e^{x\tau} |\Phi|(\mathrm{d}\tau) + \int_{-d}^{0} e^{x\tau} \cos(y\tau)\, \Phi(\mathrm{d}\tau) \\ &\le - \int_{-d}^{0} e^{x\tau}\, (1 - \cos(y\tau))\, \Phi(\mathrm{d}\tau). \end{aligned}$$

At this point, writing $\Phi = \Phi^+ - \Phi^-$, with Φ^+ and Φ^- the positive and negative part of Φ, respectively, we have

$$\begin{aligned}\tilde{K}(x) &\leq -\int_{-d}^{0} e^{x\tau}\,(1-\cos(y\tau))\,\Phi(\mathrm{d}\tau)\\ &= -\int_{-d}^{0} e^{x\tau}\,(1-\cos(y\tau))\,\Phi^+(\mathrm{d}\tau) + \int_{-d}^{0} e^{x\tau}\,(1-\cos(y\tau))\,\Phi^-(\mathrm{d}\tau) \leq 0.\end{aligned}$$

Since λ was a generic element of the spectrum of A_1 we have that $\tilde{K}(\lambda_0) \leq 0$. This concludes the proof. □

Lemma 4.3. *Let $\lambda \in \mathbf{R} \cap \rho(A_1)$. Then the resolvent $R(\lambda, A_1)$ is given by*

$$R(\lambda, A_1)\begin{pmatrix} m_0 \\ m_1 \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \tag{24}$$

with

$$\begin{aligned} u_0 &= \frac{1}{K(\lambda)}\big[m_0 + \int_{-d}^{0}\int_{-d}^{s_1} e^{\lambda(\tau-s_1)}\Phi(d\tau)\,m_1(s_1)\,ds_1\big],\\ u_1(s) &= \frac{e^{\lambda s}}{K(\lambda)}\left(m_0 + \int_{-d}^{0}\int_{-d}^{s_1} e^{\lambda(\tau-s_1)}\Phi(d\tau)\,m_1(s_1)\,ds_1\right) + \int_{s}^{0} e^{-\lambda(s_1-s)}m_1(s_1)\,ds_1. \end{aligned} \tag{25}$$

Proof. To compute $R(\lambda, A_1)$, we will consider for a fixed $\begin{pmatrix} m_0 \\ m_1 \end{pmatrix} \in \mathcal{H}$ the equation

$$(\lambda - A_1)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} m_0 \\ m_1 \end{pmatrix}, \tag{26}$$

that by definition of A_1 is equivalent to

$$\begin{cases} (\lambda - (\mu_0 - \sigma_0^\top \kappa))u_0 - \displaystyle\int_{-d}^{0} u_1(\tau)\Phi(\mathrm{d}\tau) = m_0 \\ \lambda u_1 - \dfrac{\mathrm{d}u_1}{\mathrm{d}s} = m_1. \end{cases}$$

Then

$$u_1(s) = e^{\lambda s}u_0 + \int_{s}^{0} e^{-\lambda(s_1-s)}m_1(s_1)\,\mathrm{d}s_1, \quad s \in [-d, 0],$$

and u_0 is determined by the equation

$$\big(\lambda - (\mu_0 - \sigma_0^\top \kappa)\big) u_0 = \Big[m_0 + \int_{-d}^{0} \Big(e^{\lambda\tau} u_0 + \int_{\tau}^{0} e^{-\lambda(s_1-\tau)} m_1(s_1)\, \mathrm{d}s_1 \Big) \Phi(\mathrm{d}\tau) \Big]$$

or equivalently, u_0 is given by the equation

$$K(\lambda) u_0 = m_0 + \int_{-d}^{0} \int_{-d}^{s_1} e^{\lambda(\tau - s_1)} \Phi(\mathrm{d}\tau)\, m_1(s_1) \mathrm{d}s_1,$$

with $K(\lambda)$ defined in (6). Thus for $K(\lambda) \neq 0$ the equation (26) is invertible and the result follows. □

Recall that by $S(t)$ we denote the strongly continuous semigroup generated by A_1. The following fact is well known.

Lemma 4.4. *For any λ with $\mathrm{Re}(\lambda) > \lambda_0$ we have*

$$\int_0^\infty e^{-\lambda t} S(t) \begin{pmatrix} m_0 \\ m_1 \end{pmatrix} dt = R(\lambda, A_1) \begin{pmatrix} m_0 \\ m_1 \end{pmatrix}. \tag{27}$$

Proof. Formula (27) is standard for any strongly continuous semigroup provided λ is big enough. To show that we can take $\lambda > \lambda_0$ we invoke the fact that the semigroup $S(t)$ is eventually compact, hence for the generators of the delay semigroups the growth bound and the spectral bound λ_0 coincide, see Corollary 2.5 on p. 121 of [4]. □

For $\lambda \in \mathbf{R}$ such that $K(\lambda) \neq 0$, let $\big(f(\lambda), g(\lambda)\big)$ be defined as

$$\begin{aligned} f(\lambda) &:= \frac{1}{K(\lambda)}, \\ g(\lambda, s) &:= \frac{1}{K(\lambda)} \int_{-d}^{s} e^{-\lambda(s-\tau)} \Phi(\mathrm{d}\tau). \end{aligned} \tag{28}$$

Lemma 4.5. *Fix $t_0 \geq 0$. Let $M = (M_0, M_1) \in \mathcal{H}$ be a solution to the following differential equation*

$$\begin{cases} \dfrac{dM(t)}{dt} = A_1 M(t), \\[2ex] M_0(t_0) = m_0, \\[2ex] M_1(t_0, s) = m_1(s), \quad s \in [-d, 0). \end{cases} \tag{29}$$

with $(m_0, m_1) \in \mathbf{R} \times L^2([-d,0];\mathbf{R})$. *Then for any* $\lambda \in \mathbb{R}$, $\lambda > \lambda_0$ *we have*

$$\int_{t_0}^{+\infty} e^{-\lambda t} M_0(t)\, dt = e^{-\lambda t_0} \langle (f(\lambda), g(\lambda,\cdot)), (m_0, m_1)\rangle_{\mathcal{H}}.$$

Proof. We first prove the result for $t_0 = 0$. Recalling Lemma 4.3 and Lemma 4.4, we have

$$\begin{aligned}\int_0^\infty e^{-\lambda t} M_0(t)\mathrm{d}t = \int_0^\infty e^{-\lambda t} S(t) m_0 \,\mathrm{d}t &= R(\lambda, A_1) m_0 \\ &= \frac{1}{K(\lambda)} \left[m_0 + \int_{-d}^0 \int_{-d}^{s_1} e^{\lambda(\tau - s_1)} \Phi(\mathrm{d}\tau)\, m_1(s_1) \mathrm{d}s_1 \right] \\ &= \langle (f(\lambda), g(\lambda,\cdot)), (m_0, m_1)\rangle_{\mathcal{H}}.\end{aligned} \tag{30}$$

Now, consider $t_0 \geq 0$, and let $\big(M_0(t;t_0,m_0,m_1), M_1(t;t_0,m_0,m_1)\big)$ be a solution to equation (29) starting at time t_0 from (m_0, m_1). Then we have

$$M_0(t;t_0,m_0,m_1) = M_0(t-t_0;0,m_0,m_1)\,.$$

By (30), it holds

$$\begin{aligned}&\int_{t_0}^{+\infty} e^{-\lambda t} M_0(t;t_0,m_0,m_1)\mathrm{d}t = \int_0^{+\infty} e^{-\lambda(s+t_0)} M_0(s;0,m_0,m_1)\mathrm{d}s \\ &= e^{-\lambda t_0} \int_0^{+\infty} e^{-\lambda s} M_0(s;0,m_0,m_1)\mathrm{d}s = e^{-\lambda t_0} \langle (f(\lambda), g(\lambda)), (m_0, m_1)\rangle_{\mathcal{H}}.\end{aligned}$$

□

In order to prove Theorem 2.1 we also need the following technical lemma.

Lemma 4.6. *It holds that*

$$\mathbf{E}\Big(\int_{t_0}^t \left\| X_0(s)\sigma_0 + \begin{pmatrix} \int_{-d}^0 X_0(s+\tau)\varphi_1(d\tau) \\ \vdots \\ \int_{-d}^0 X_0(s+\tau)\varphi_n(d\tau) \end{pmatrix} \right\|_{\mathbf{R}^n}^2 ds \Big) < +\infty\,.$$

Proof. Let us denote with σ_0^i the i-th component of σ_0, and let us show that

$$\mathbf{E}\Big(\int_{t_0}^t \big[X_0(s)\sigma_0^i + \int_{-d}^0 X_0(s+\tau)\varphi_i(\mathrm{d}\tau) \big]^2 \mathrm{d}s \Big) < +\infty.$$

By the trivial inequality $(a+b)^2 \le 2(a^2+b^2)$, it is sufficient to show that

$$\mathbf{E}\Big(\int_{t_0}^{t} X_0^2(s)(\sigma_0^i)^2 \mathrm{d}s\Big) < +\infty, \tag{31}$$

and

$$\mathbf{E}\Big(\int_{t_0}^{t} \Big[\int_{-d}^{0} X_0(s+\tau)\varphi_i(\mathrm{d}\tau)\Big]^2 \mathrm{d}s\Big) < +\infty. \tag{32}$$

To show (31), by Theorem 7.4 in [12] we can write

$$\mathbf{E}\Big(\int_{t_0}^{t} X_0^2(s)(\sigma_0^i)^2 \mathrm{d}s\Big) \le (\sigma_0^i)^2(t-t_0)\mathbf{E}\Big(\sup_{s\in[t_0,t]} X_0^2(s)\Big) < +\infty.$$

To show (32), by the Hölder inequality

$$\begin{aligned}\Big(\int_{-d}^{0} X_0(s+\tau)\,\varphi_i(\mathrm{d}\tau)\Big)^2 &\le \Big(\int_{-d}^{0} |X_0(s+\tau)|^2\,\varphi_i(\mathrm{d}\tau)\Big)\Big(\int_{-d}^{0} \varphi_i(\mathrm{d}\tau)\Big)\\ &= \varphi_i([-d,0])\Big(\int_{-d}^{0} |X_0(s+\tau)|^2\,\varphi_i(\mathrm{d}\tau)\Big).\end{aligned}$$

Thus

$$\begin{aligned}\mathbf{E}\Big(\int_{t_0}^{t} \Big[\int_{-d}^{0} X_0(s+\tau)\varphi_i(\mathrm{d}\tau)\Big)^2 \mathrm{d}s\Big] &\le \varphi_i([-d,0])\int_{t_0}^{t}\int_{-d}^{0} \mathbf{E}\Big(|X_0(s+\tau)|^2\Big)\,\varphi_i(\mathrm{d}\tau)\mathrm{d}s\\ &\le \Big(\varphi_i([-d,0])\Big)^2 (t-t_0)\sup_{\tau\in[-d,0]}\sup_{s\in[t_0,t]} \mathbf{E}\Big(|X_0(s+\tau)|^2\Big).\end{aligned}$$

By Theorem 7.4 in [12], the expression above is finite. □

We can now provide the proof of Theorem 2.1.

Proof. We have

$$\mathbf{E}\Big(\int_{t_0}^{+\infty} \xi(s)X_0(s)\mathrm{d}s \mid \mathcal{F}_{t_0}\Big) = \int_{t_0}^{+\infty} \mathbf{E}\big(\xi(s)X_0(s) \mid \mathcal{F}_{t_0}\big)\mathrm{d}s \quad \mathbf{P}\text{-a.s.} \tag{33}$$

In fact, using the characteristic property of the conditional mean, and Fubini's Theorem together with Theorem 7.4 in [12], for any $F \in \mathcal{F}_{t_0}$ we have

$$\int_F \mathbf{E}\Big(\int_{t_0}^{+\infty} \xi(s)X_0(s)\mathrm{d}s \mid \mathcal{F}_{t_0}\Big)\mathrm{d}\mathbf{P} = \int_F \int_{t_0}^{+\infty} \xi(s)X_0(s)\mathrm{d}s\,\mathrm{d}\mathbf{P}$$
$$= \int_{t_0}^{+\infty}\int_F \xi(s)X_0(s)\mathrm{d}\mathbf{P}\,\mathrm{d}s = \int_{t_0}^{+\infty}\int_F \mathbf{E}\big(\xi(s)X_0(s) \mid \mathcal{F}_{t_0}\big)\mathrm{d}\mathbf{P}\,\mathrm{d}s$$
$$= \int_F \int_{t_0}^{+\infty} \mathbf{E}\big(\xi(s)X_0(s) \mid \mathcal{F}_{t_0}\big)\mathrm{d}s\,\mathrm{d}\mathbf{P}.$$

To compute $\mathbf{E}\big(\xi(s)X_0(s) \mid \mathcal{F}_{t_0}\big)$, let us consider the equivalent probability measure

$$\mathrm{d}\tilde{\mathbf{P}}(s) := e^{-\frac{1}{2}|\kappa|^2 s - \kappa^\top Z_s}\mathrm{d}\mathbf{P}\,,$$

defined on $\mathcal{F}_s$. Note that

$$\frac{\mathrm{d}\tilde{\mathbf{P}}(s)}{\mathrm{d}\mathbf{P}} = e^{-\frac{1}{2}|\kappa|^2 s - \kappa^\top Z_s} = e^{rs}\xi(s)\,,$$

and hence by Lemma 3.5.3 in [23] we can write

$$\mathbf{E}\Big(\xi(s)X_0(s) \mid \mathcal{F}_{t_0}\Big) = \xi(t_0)e^{-r(s-t_0)}\tilde{\mathbf{E}}\Big(X_0(s) \mid \mathcal{F}_{t_0}\Big),$$

where $\tilde{\mathbf{E}}$ denotes the mean under the measure $\tilde{\mathbf{P}}(s)$. Our aim is to evaluate

$$\int_{t_0}^{+\infty} \mathbf{E}\Big(\xi(s)X_0(s) \mid \mathcal{F}_{t_0}\Big)\mathrm{d}s = \xi(t_0)e^{rt_0}\int_{t_0}^{+\infty} e^{-rs}\tilde{\mathbf{E}}\Big(X_0(s) \mid \mathcal{F}_{t_0}\Big)\mathrm{d}s. \tag{34}$$

Let $\tilde{\mathbf{P}}$ denote the measure, such that $\tilde{\mathbf{P}}\Big|_{\mathcal{F}_s} = \tilde{\mathbf{P}}(s)$ for all $s \geq 0$. By the Girsanov Theorem, the process

$$\tilde{Z}(t) = Z(t) + \kappa t \tag{35}$$

is an n-dimensional Brownian motion under the measure $\tilde{\mathbf{P}}$, and the dynamics of X_0 under $\tilde{\mathbf{P}}$ is

$$dX_0(s) = \quad \big[(\mu_0 - \sigma_0^\top\kappa)X_0(s) + \textstyle\int_{-d}^0 X_0(s+\tau)\,\Phi(\mathrm{d}\tau)\big]\mathrm{d}s$$
$$+ \left[X_0(t)\sigma_0^\top + \begin{pmatrix} \int_{-d}^0 X_0(s+\tau)\varphi_1(\mathrm{d}\tau) \\ \vdots \\ \int_{-d}^0 X_0(s+\tau)\varphi_n(\mathrm{d}\tau)\end{pmatrix}^\top\right]\mathrm{d}\tilde{Z}(s),$$

where Φ is defined in (8). Integrating between t_0 and t we obtain

$$\begin{aligned}X_0(t) = X_0(t_0) &+ \int_{t_0}^{t} (\mu_0 - \sigma_0^\top \kappa) X_0(s) \mathrm{d}s + \int_{t_0}^{t} \int_{-d}^{0} X_0(s+\tau)\Phi(\mathrm{d}\tau)\mathrm{d}s \\ &+ \int_{t_0}^{t} \left[X_0(s)\sigma_0^\top + \begin{pmatrix} \int_{-d}^{0} X_0(s+\tau)\varphi_1(\mathrm{d}\tau) \\ \vdots \\ \int_{-d}^{0} X_0(s+\tau)\varphi_n(\mathrm{d}\tau) \end{pmatrix}^\top \right] \mathrm{d}\tilde{Z}(s)\,,\end{aligned} \tag{36}$$

and therefore

$$\begin{aligned}\tilde{\mathbf{E}}\Big(X_0(t) \mid \mathcal{F}_{t_0}\Big) =& X_0(t_0) + (\mu_0 - \sigma_0^\top \kappa)\tilde{\mathbf{E}}\Big(\int_{t_0}^{t} X_0(s)\mathrm{d}s \mid \mathcal{F}_{t_0}\Big) \\ &+ \tilde{\mathbf{E}}\Big(\int_{t_0}^{t} \int_{-d}^{0} X_0(s+\tau)\Phi(\mathrm{d}\tau)\mathrm{d}s \mid \mathcal{F}_{t_0}\Big) \\ &+ \tilde{\mathbf{E}}\left(\int_{t_0}^{t} \left[X_0(s)\sigma_0^\top + \begin{pmatrix} \int_{-d}^{0} X_0(s+\tau)\varphi_1(\mathrm{d}\tau) \\ \vdots \\ \int_{-d}^{0} X_0(s+\tau)\varphi_n(\mathrm{d}\tau) \end{pmatrix}^\top \right] \mathrm{d}\tilde{Z}(s) \mid \mathcal{F}_{t_0} \right).\end{aligned} \tag{37}$$

By Lemma 4.6, which still applies after the change of measure, the stochastic integral with respect to $\tilde{Z}$ is a martingale, and has zero mean. By definition of conditional mean and by Fubini's Theorem, the expression in (37) gives

$$\begin{aligned}\tilde{\mathbf{E}}\Big(X_0(t) \mid \mathcal{F}_{t_0}\Big) =& X_0(t_0) + (\mu_0 - \sigma_0^\top \kappa) \int_{t_0}^{t} \tilde{\mathbf{E}}\big(X_0(s) \mid \mathcal{F}_{t_0}\big)\,\mathrm{d}s \\ &+ \int_{t_0}^{t} \int_{-d}^{0} \tilde{\mathbf{E}}\big(X_0(s+\tau) \mid \mathcal{F}_{t_0}\big)\,\Phi(\mathrm{d}\tau)\mathrm{d}s.\end{aligned} \tag{38}$$

Deriving (38) with respect to t, we obtain the following, for $t > t_0$:

$$\frac{\mathrm{d}\tilde{\mathbf{E}}\Big(X_0(t) \mid \mathcal{F}_{t_0}\Big)}{\mathrm{d}t} = (\mu_0 - \sigma_0^\top \kappa)\tilde{\mathbf{E}}\Big(X_0(t) \mid \mathcal{F}_{t_0}\Big) + \int_{-d}^{0} \tilde{\mathbf{E}}\Big(X_0(t+\tau) \mid \mathcal{F}_{t_0}\Big)\,\Phi(\mathrm{d}\tau). \tag{39}$$

We then see that $\tilde{\mathbf{E}}\big(X_0(t) \mid \mathcal{F}_{t_0}\big)$ must be a solution of

$$\begin{cases} \dfrac{\mathrm{d}M_0}{\mathrm{d}t}(t) = (\mu_0 - \sigma_0^\top \kappa) M_0(t) + \int_{-d}^{0} M_0(t+s)\,\Phi(\mathrm{d}s), & t > 0, \\ M_0(t_0) = m_0, & \\ M_1(t_0, s) = m_1(s), & s \in [-d, 0). \end{cases} \tag{40}$$

By Hypothesis 1 and Lemmas 4.1 and 4.2 we have $r > \lambda_0$, hence invoking Lemma 4.5 we obtain

$$\int_{t_0}^{+\infty} e^{-rt} \tilde{\mathbf{E}}\big(X_0(t) \mid \mathcal{F}_{t_0}\big) \mathrm{d}t = e^{-rt_0} \langle \big(f(r), g(r, \cdot)\big), (m_0, m_1) \rangle_{\mathcal{H}}.$$

Recalling (33) and (34), we can write

$$\begin{aligned} \mathbf{E}\Big(\int_{t_0}^{+\infty} \xi(s) X_0(s) \mathrm{d}s \mid \mathcal{F}_{t_0} \Big) &= \xi(t_0) e^{rt_0} \int_{t_0}^{+\infty} e^{-rs} \tilde{\mathbf{E}}\Big(X_0(s) \mid \mathcal{F}_{t_0}\Big) \mathrm{d}s \\ &= \xi(t_0) \langle \big(f(r), g(r, \cdot)\big), (m_0, m_1) \rangle_{\mathcal{H}}. \end{aligned}$$

Note that $\big(f(r), g(r, \cdot)\big) = (\frac{1}{K}, \frac{1}{K} G(\cdot))$, with (f, g) defined in (28), K in (9), and G in (13). The proof is thus complete. $\square$

5 Conclusion

In this paper, we have provided a valuation formula for streams of payments with delayed dynamics in an otherwise standard, complete market model with risky assets driven by a GBM. As a practical example, we have discussed the application of our analysis to the valuation of human capital in a setting with sticky wages, where wage rigidity is obtained by introducing delay terms in the drift and volatility coefficients of an otherwise standard GBM labor income dynamics. Our valuation formula results in an explicit expression of human capital demonstrating the importance of appreciating the past to quantify the current market value of future labor income. More generally, the approach followed in this paper shows how tools from infinite-dimensional analysis can be successfully used to address valuation problems that are non-Markovian, and hence beyond the reach of coventional approaches.

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