THE FUNDAMENTAL GAP FOR A ONE-DIMENSIONAL SCHRÖDINGER OPERATOR WITH ROBIN BOUNDARY CONDITIONS

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Abstract. For Schrödinger operators on an interval with either convex or symmetric single-well potentials, and Robin or Neumann boundary conditions, the gap between the two lowest eigenvalues is minimised when the potential is constant. We also have results for the p-Laplacian.

1. Introduction

In studying the eigenvalues of a differential operator, one important quantity is the gap between the first and second eigenvalues, called the *fundamental gap*. This is of both physical and mathematical importance: in the context of the heat equation, it gives the rate of collapse of any initial state to the ground state; computationally, it can control the rate of convergence of a numerical scheme [12].

The fundamental gap for the classical Schrödinger operator $-\Delta + V$ has been extensively studied under *Dirichlet* boundary values. In one dimension, lower bounds of this gap were found under various assumptions on V [3, 7] until Lavine [9] found the sharp result: the gap for a convex potential is minimised by the gap for a constant potential. The analogous result in higher dimensions, on a convex domain, was resolved several years later [1]. Smits considered the question of the lower bounds on the fundamental gap under *Robin* boundary conditions [13], however there are very few results known in this case. Laugesen recently studied the Robin eigenvalues, and the gap, on rectangles [8]. The Robin problem is much more sensitive to the boundary, and is thus more difficult. For example, the method used in [1] to prove sharp lower bounds on the Dirichlet fundamental gap uses the property that the first Dirichlet eigenfunction is *log-concave*: in recent work, we have shown that the first Robin eigenfunction does not always enjoy that property [2].

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Thus, one important motivation here is to find methods to derive sharp lower bounds on the gap that do not rely on the log-concavity of the first eigenfunction. In one dimension, Lavine's proof of the lower bound for the gap is such a method.

In this paper, we revisit Lavine's method and establish sharp lower bounds of the fundamental gap for the classical (linear) Schrödinger operator on a bounded interval under Robin boundary conditions. We not only deal with the case where the Robin parameter α is positive, but also for $-\frac{1}{2} \leq \alpha \leq 0$, thus also including the Neumann case. The same statements also hold for the Dirichlet case (sometimes referred to as $\alpha = \infty$). We also extend some results to the nonlinear Schrödinger operator associated with the p-Laplace operator.

We can extend the methods to Robin boundary conditions largely because the boundary conditions generally appear in forms such as $[u_1 u_0' - u_1' u_0]_1^{-1}$ and so they not only vanish in the Neumann or Dirichlet cases, but also in the Robin case.

We now introduce our notation and assumptions used through this paper. Let I denote the open interval $(-1, 1)$ and V a potential function in $C(\overline{I})$. The eigenvalue problem for the classical Schrödinger operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V$ on I is to find eigenpairs (u_i, λ_i^V) solving

$$-u'' + Vu = \lambda^V u \quad \text{on } I \tag{1.1}$$

subject to either homogeneous *Robin or Neumann boundary conditions*

$$u'(\pm 1) = \mp \alpha u(\pm 1), \tag{1.2}$$

or homogeneous *Dirichlet boundary condition*

$$u(\pm 1) = 0. \tag{1.3}$$

The boundary conditions (1.2) are called *Neumann* if $\alpha = 0$, and otherwise *Robin* with *Robin parameter* $\alpha \in \mathbb{R}$. Dividing (1.2) by $\alpha > 0$ and sending $\alpha \to +\infty$, one recovers the *Dirichlet* conditions (1.3).

For $1 < p < +\infty$, we also consider the Robin eigenvalue problem for the nonlinear Schrödinger operator $-\Delta_p + V|\cdot|^{p-2}\cdot$ associated with the p-Laplace operator $\Delta_p u := (|u'|^{p-2}u')'$. Here we seek to find eigenpairs (u_i, λ_i^V) solving

$$-\Delta_p u + V|u|^{p-2}u = \lambda^V |u|^{p-2}u \quad \text{on } I, \tag{1.4}$$

with Robin boundary conditions

$$|u'|^{p-2}u' = \mp \alpha |u|^{p-2}u \text{ at } x = \pm 1. \tag{1.5}$$

The Dirichlet boundary condition is again (1.3).

For $p = 2$ equations (1.4) and (1.5) reduce to the classical linear ones (1.1) and (1.2). For given $V \in L^1(I)$, $\lambda \in \mathbb{R}$, and $\alpha \in \mathbb{R}$, we call a function $u \in C(\overline{I})$ a *solution* of equation (1.4) satisfying (1.5) provided $u \in W^{1,p}(I)$ and satisfies

$$\int_I |u'|^{p-2}u'\xi' + \big(V-\lambda\big)|u|^{p-2}u\,\xi\;\mathrm{d}x + \alpha\Big[(|u|^{p-2}u\,\xi)_{|x=1} + (|u|^{p-2}u\,\xi)_{|x=-1}\Big] = 0$$

for every $\xi \in C^1(\bar{I})$. Since for every solution u of (1.4), $\lambda^V - V\,|u|^{p-2}u \in L^1(I)$, one has that $|u'|^{p-2}u' \in W^{1,1}(I)$ and since $W^{1,1}(I)$ is continuously embedded into $C(\bar{I})$, one finds that $u' \in C(\bar{I})$. Thus, every solution u of (1.4) has regularity $u \in C^1(\bar{I})$.

The Ljusternik-Schnirelmann theory [6, 10] ensures the existence of a sequence $(\lambda_i^V, u_i)_{i\geq 0}$ of eigenpairs (λ_i^V, u_i) for the nonlinear Schrödinger operator $-\Delta_p + V|\cdot|^{p-2}\cdot$ with Robin boundary conditions (1.5). For every $i \in \mathbb{N}$,

$$\lambda_i^V := \inf_{W\in\Gamma_i}\max_{u\in W}\mathcal{R}[u,V,p,\alpha], \tag{1.6}$$

where $\mathcal{R}$ is the *Rayleigh quotient*

$$\mathcal{R}[u,V,p,\alpha] = \frac{\displaystyle\int_I |u'|^p + V|u|^p\;\mathrm{d}x + \alpha\,[|u|^p(1) + |u|^p(-1)]}{\displaystyle\int_I |u|^p\;\mathrm{d}x}. \tag{1.7}$$

In (1.6), the maximum is attained among all $u \in W \subseteq \Gamma_i$, where Γ_i is a specific closed subset of $W^{1,p}(I)$. To obtain eigenpairs (λ_i^V, u_i) for homogeneous Neumann boundary conditions, one chooses $\alpha = 0$, while for homogeneous Dirichlet boundary conditions (1.3), one needs to replace the space $W^{1,p}(I)$ by $W^{1,p}_0(I)$.

Our primary object of interest in this paper is the fundamental gap

$$\Gamma_p(V) := \lambda_1^V - \lambda_0^V.$$

We write $\Gamma_p(0)$ for the fundamental gap of the zero potential $V \equiv 0$. A potential V is called *single-well* if there is a point $x_0 \in I$ such that V is non-increasing along $(-1, x_0)$ and non-decreasing along $(x_0, 1)$. A potential V on I is called *symmetric* if $V(-x) = V(x)$.

Our first result concerns the fundamental gap for such symmetric, single-well potentials. The analogous theorem with Dirichlet boundary conditions, and $p = 2$, is due to Ashbaugh and Benguria [3].

Theorem 1.1. *For $\alpha \in \mathbb{R}$, consider the nonlinear eigenvalue problem* (1.4) *with Robin boundary conditions* (1.5). *Then for every symmetric, single well potential V, the fundamental gap satisfies*

$$\Gamma_p(V) \geq \Gamma_p(0)$$

with equality only when V is constant.

Our next theorem shows that the fundamental gap $\Gamma_p(V)$ with convex potential V is minimised by a linear potential.

Theorem 1.2. *For $\alpha \in \mathbb{R}$, consider the nonlinear eigenvalue problem* (1.4) *with Robin boundary conditions* (1.5). *Then for every convex potential V which is not affine, there exists a linear potential $V_a = ax$, such that*

$$\Gamma_p(V) > \Gamma_p(ax).$$

The last two theorems relate to the classical, $p = 2$, case only.

Theorem 1.3. *Consider the linear eigenvalue problem* (1.1) *with Robin boundary conditions* (1.5) *with $\alpha \geq -\frac{1}{2}$, and let $V_a = ax$ be a linear potential, $a \in \mathbb{R}$. Then the fundamental gap is bounded below by the gap for a zero potential,*

$$\Gamma_2(ax) \geq \Gamma_2(0).$$

with equality only when $a = 0$.

Combining Theorem 1.2 with Theorem 1.3, we immediately have:

Theorem 1.4. *Consider the linear eigenvalue problem* (1.1) *where V is a* convex potential, *and with Robin boundary conditions* (1.2) *with $\alpha \geq -\frac{1}{2}$. Then the fundamental gap satisfies*

$$\Gamma_2(V) \geq \Gamma_2(0).$$

In the linear case $p = 2$ and under homogeneous Dirichlet and Neumann boundary conditions, Theorems 1.2, 1.3 and 1.4 are due to Lavine [9].

The structure of this paper is as follows. In Section 2 we establish some necessary technical results, particularly about the shape of the first two eigenfunctions. In Section 3, we prove Theorem 1.1, for symmetric single-well potentials. In Section 4, we prove Theorem 1.2. In Section 5, we prove some further technical results to support the argument in Section 6, where we prove Theorem 1.3: this proof is only for the *classical* Schrödinger operator, with $p = 2$.

2. Some preliminary results

We normalise all eigenfunctions so that $\int_I |u_i|^p \,\mathrm{d}x = 1$ and $u_i > 0$ on $(-1, -1 + \epsilon)$ for some small $\epsilon > 0$. If (u_0, λ_0^V) is the first eigenpair then u_0 minimises the Rayleigh quotient (1.7). Since also $|u_0|$ is a minimiser of (1.7), we can choose $u_0 \geq 0$ on $\overline{I}$. But as we assume that the potential V is bounded from below on I, and since u_0 is a solution of (1.4), the strong maximum principle (see [11, Theorem 5.3.1]) implies that $u_0 > 0$ on I.

We begin with a Hellmann–Feynmann result for the variation of eigenvalues with respect to a family of potentials.

Lemma 2.1. *Let $\{V^t\}_{t\in J}$ be a family of potentials $V^t \in L^1(I)$ varying differentiably in the parameter $t \in \mathbb{R}$. Then for every eigenpair (u_i, λ_i^V) of the nonlinear eigenvalue problem* (1.4) *with Robin boundary conditions* (1.5)*,*

$$\frac{\partial}{\partial t}\lambda_i^{V^t} = \int_I \dot{V}^t |u_i|^p \, dx$$

where $\dot{V}^t$ indicates the derivative $\frac{\partial}{\partial t}V^t$ with respect to t. In particular, for the fundamental gap,

$$\frac{\partial}{\partial t}\Gamma_p(V^t) = \int_I \dot{V}^t \left(|u_1|^p - |u_0|^p\right) \, dx. \tag{2.1}$$

Proof. By using the Rayleigh quotient (1.7) and the fact that (u_i, λ_i^V) is an eigenpair of (1.4) satisfying Robin boundary conditions (1.5),

$$\begin{aligned}
\frac{\partial}{\partial t}\lambda_i^{V^t} &= \frac{1}{\int_I |u_i|^p \,\mathrm{d}x}\Bigg[\int_I p|u_i'|^{p-2}u_i'\dot{u}_i' + \dot{V}^t|u_i|^p + V^t p|u_i|^{p-2}u_i\dot{u}_i \,\mathrm{d}x \\
&\qquad\qquad + \alpha\Big[p|u_i|^{p-2}u_i\dot{u}_i(-1) + p|u_i|^{p-2}u_i\dot{u}_i(1)\Big]\Bigg] \\
&\qquad - \frac{1}{\left(\int_I |u_i|^p \,\mathrm{d}x\right)^2}\Bigg[\int_I |u_i'|^p + V^t|u_i|^p \,\mathrm{d}x \\
&\qquad\qquad + \alpha\Big[|u_i|^p(-1) + |u_i|^p(1)\Big]\Bigg]\left[\int_I p|u_i|^{p-2}u_i\dot{u}_i \,\mathrm{d}x\right] \\
&= \frac{1}{\int_I |u_i|^p \,\mathrm{d}x}\left[\lambda_i^{V^t}\int_I p|u_i|^{p-2}u_i\dot{u}_i \,\mathrm{d}x + \int_I \dot{V}^t|u_i|^p \,\mathrm{d}x\right] \\
&\qquad - \frac{\lambda_i^{V^t}}{\int_I |u_i|^p \,\mathrm{d}x}\int_I p|u_i|^{p-2}u_i\dot{u}_i \,\mathrm{d}x \\
&= \frac{1}{\int_I |u_i|^p \,\mathrm{d}x}\int_I \dot{V}^t|u_i|^p \,\mathrm{d}x.
\end{aligned}$$

□

For a given $\lambda \in \mathbb{R}$, let u be a solution of the nonlinear eigenvalue problem (1.4)-(1.5). Let v be defined by

$$v = \frac{|u'|^{p-2}u'}{|u|^{p-2}u} \qquad \text{on } I. \tag{2.2}$$

Then v is a solution of the Riccati equation

$$v' = (V - \lambda) - (p-1)\,|v|^{\frac{p}{p-1}} \tag{2.3}$$

on I, and by (1.5), satisfies the inhomogeneous Dirichlet boundary conditions

$$v(\pm 1) = \mp\alpha. \tag{2.4}$$

For every $\lambda \in \mathbb{R}$, the function

$$f_\lambda(x, v) := (V - \lambda) - (p-1)\,|v|^{\frac{p}{p-1}} \qquad \text{for a.e. } x \in \bar{I} \text{ and all } v \in \mathbb{R},$$

has a continuous partial derivative $\frac{\partial}{\partial v} f_\lambda(x, v) = -p|v|^{\frac{2-p}{p-1}} v$, uniformly for a.e. $x \in I$. Thus Gronwall's lemma implies that for every $c \in \mathbb{R}$ and $x_0 \in \bar{I}$, there can be at most one bounded solution v on $\bar{I}$ of (2.3) satisfying the initial condition $v(x_0) = c$.

The first eigenfunction is strictly positive and so the corresponding solution v_0 of (2.3) is bounded on $\bar{I}$ and hence is unique. By (2.2), this means the first eigenpair (u_0, λ_0^V) is *simple*, in the sense that any two solutions of (1.4)-(1.5) for the same eigenvalue are linearly dependent.

Concerning the simplicity of the other eigenpairs, a generalisation of *Courant's nodal domain theorem* for the p-Laplace operator with homogeneous Dirichlet boundary conditions on a bounded smooth domain in $\mathbb{R}^d$ was obtained by Drábek and Robinson [5]. A *nodal domain* is defined as a maximal connected open subset $\{u(x) \neq 0\}$. In one dimension, a Sturm-Liouville theory for nonlinear Schrödinger operators of the form $-\Delta_p + V|\cdot|^{p-2}\cdot$ on the bounded interval $(0, b)$ satisfying $u'(0) = 0$ and Robin boundary conditions at the right endpoint $x = b$ was elaborated by several authors (e.g. [4] or [14, Theorem 5 & subsequent Corollary]). The results and techniques of the last two references imply that a nodal domain theorem for $-\Delta_p + V|\cdot|^{p-2}\cdot$ with Robin boundary condition holds. We omit the details.

Lemma 2.2 (Nodal domain theorem for Robin boundary conditions)**.** *Let $V \in C(\bar{I})$ and $\alpha \in \mathbb{R}$. Then each eigenvalue λ_i^V of the Schrödinger operator $-\Delta_p + V|\cdot|^{p-2}\cdot$ with Robin boundary conditions* (1.5) *is simple, satisfies*

$$\lambda_0^V < \lambda_1^V < \lambda_2^V < \cdots \to \infty.$$

Moreover, the corresponding eigenvector u_i has exactly $i+1$ nodal domains in I.

With the help of the preceding lemma, we obtain the following monotonicity property.

Lemma 2.3. *For $\alpha \in \mathbb{R}$, let (u_0, λ_0^V) and (u_1, λ_1^V) be the first and second eigenpair of the Schrödinger operator $-\Delta_p + V|\cdot|^{p-2}\cdot$ with Robin boundary conditions* (1.5)*. Then the ratio u_1/u_0 is monotonically decreasing in x.*

Proof. By the nodal domain theorem (Lemma 2.2), u_1 admits exactly one zero x_0 in I, and by construction, u_1 is positive near $x = -1$. Thus

$$\phi := \log \frac{u_1}{u_0}$$

is well-defined on $[-1, x_0)$. We claim that $\phi' < 0$ on $[-1, x_0)$. We compute

$$\phi' = \frac{u_1'}{u_1} - \frac{u_0'}{u_0} = |v_1|^{p^\star} v_1 - |v_0|^{p^\star} v_0,$$

where v_i is defined by (2.2) for $u = u_i$ and $p^\star = -(p-2)/(p-1)$. As $s \mapsto s|s|^{p^\star}$ is increasing, $\phi' < 0$ on $[-1, x_0)$ if $v_1 < v_0$ on $[-1, x_0)$.

By (2.4), $v_0(-1) = \alpha = v_1(-1)$ and since $\lambda_0^V < \lambda_1^V$, the Riccati equation (2.3) implies that

$$v_1'(-1) = (V - \lambda_1^V) - (p-1)|\alpha|^{\frac{p}{p-1}} < (V - \lambda_0^V) - (p-1)|\alpha|^{\frac{p}{p-1}} = v_0'(-1).$$

Hence $v_1 < v_0$ in a neighbourhood of -1.

Since $v_i \in C(\bar{I})$, there exists a largest $y_0 \in (-1, x_0]$ such that $v_1 < v_0$ on $(-1, y_0)$.

If $y_0 < x_0$, then $v_1(y_0) = v_0(y_0)$. The Riccati equation (2.3), this time evaluated at y_0, implies that $v_1'(\xi_0) < v_0'(\xi_0)$ and hence $v_1 > v_0$ on $(y_0 - \epsilon, y_0)$, contradicting that $v_1 < v_0$ on $(-1, y_0)$. Therefore $y_0 = x_0$, $v_1 < v_0$, and $\phi' < 0$ on $[-1, x_0)$.

Similarly, one can show that $\log(-u_1/u_0)$ is increasing on $(x_0, 1]$. It follows that the ratio u_1/u_0 is monotonically decreasing on the whole interval $\bar{I}$. □

Lemma 2.4. *Let (u_0, λ_0^V) and (u_1, λ_1^V) be the first and second eigenpair of the Schrödinger operator $-\Delta_p + V|\cdot|^{p-2}\cdot$ with Robin boundary conditions* (1.5)*. Then $|u_1|^p - |u_0|^p$ has at least one and at most two zeroes in I. To be precise, there exists ξ_-, $\xi_+ \in \bar{I}$ at least one of which is an interior point of I, such that $\xi_- < \xi_+$, and*

$$\begin{aligned} &|u_1|^p - |u_0|^p < 0 \text{ on } (\xi_-, \xi_+) \quad \text{and} \\ &|u_1|^p - |u_0|^p > 0 \text{ on } I \setminus [\xi_-, \xi_+]. \end{aligned} \tag{2.5}$$

Proof. The normalised eigenfunctions satisfy $\int_I |u_0|^p \, \mathrm{d}x = \int_I |u_1|^p \, \mathrm{d}x = 1$, and so

$$\psi := |u_1|^p - |u_0|^p$$

has mean value $⨍_I \psi \, \mathrm{d}x = 0$. At the zero x_0 of u_1, $\psi(x_0) = -|u_0|^p(x_0) < 0$, and so there must be a $y_0 \in I$ such that $\psi(y_0) > 0$. Since ψ changes sign in I, it has at least one zero ξ_0. On the other hand, ξ is a zero of ψ if and only if the ratio $\frac{u_1}{u_0}(\xi) = \pm 1$. By Lemma 2.3, u_1/u_0 is monotonically decreasing, so there exists at most two points $-1 \le \xi_- < \xi_+ \le 1$ satisfying $\frac{u_1}{u_0}(\xi_-) = 1$ and $\frac{u_1}{u_0}(\xi_+) = -1$. One of these is ξ_0. Therefore, ψ has at least one zero in I and at most two zeroes in $\bar{I}$. □

3. Fundamental gap estimates for symmetric single-well potentials

In this section, we prove Theorem 1.1 using Lemma 2.1–Lemma 2.4.

Proof of Theorem 1.1. Let V be a symmetric single-well potential. Define $\{V^t\}_{t\in\mathbb{R}^+}$ by $V^t(x) := tV(x)$. For this family of potentials, $\dot{V}^t = V$ and so (2.1) results in

$$\frac{\partial}{\partial t}\Gamma_p(V^t) = \int_I V \left(|u_1|^p - |u_0|^p\right) \mathrm{d}x$$

where (u_0, λ_0^V) and (u_1, λ_1^V) are the first and second eigenpairs of the Schrödinger operator $-\Delta_p + V|\cdot|^{p-2}\cdot$ with Robin boundary conditions (1.5).

As V is symmetric on $\bar{I}$, the eigenfunctions u_0 and u_1 are symmetric and antisymmetric respectively. Hence $|u_1|^p - |u_0|^p$ is symmetric, and the two zeroes of $|u_1|^p - |u_0|^p$ found in Lemma 2.4 are likewise symmetric, with $\xi_- = -\xi_+$.

Since V is symmetric and single-well, V is non-increasing on $(-1, 0)$ and non-decreasing on $(0, 1)$. Thus, using (2.5), we have

$$\begin{aligned}
\frac{\partial}{\partial t}\Gamma_p(V^t) &= \int_{-1}^{1} V\left(|u_1|^p - |u_0|^p\right)\,\mathrm{d}x \\
&= \int_{-1}^{\xi_-} V\left(|u_1|^p - |u_0|^p\right)\,\mathrm{d}x + \int_{\xi_-}^{\xi_+} V\left(|u_1|^p - |u_0|^p\right)\,\mathrm{d}x \\
&\qquad + \int_{\xi_+}^{1} V\left(|u_1|^p - |u_0|^p\right)\,\mathrm{d}x \\
&\geq V(\xi_-)\int_{-1}^{\xi_-} \left(|u_1|^p - |u_0|^p\right)\,\mathrm{d}x + V(\xi_\pm)\int_{\xi_-}^{\xi_+} \left(|u_1|^p - |u_0|^p\right)\,\mathrm{d}x \\
&\qquad + V(\xi_+)\int_{\xi_+}^{1} \left(|u_1|^p - |u_0|^p\right)\,\mathrm{d}x \\
&= V(\xi_\pm)\int_I \left(|u_1|^p - |u_0|^p\right)\mathrm{d}x = 0
\end{aligned}$$

with equality only when V is constant. Summarising, we have shown that

$$\frac{\partial}{\partial t}\Gamma_p(V^t) \geq 0$$

with equality only when V is constant. Integrating with respect to t over $(0, 1)$ proves the theorem. $\square$

4. Comparison of the fundamental gap between convex and linear potentials

Using a similar strategy to that used in the previous section, we can show that the fundamental gap among convex potentials is minimised by a linear one.

Proof of Theorem 1.2. Let V be a convex potential on I which is not affine. Let ξ_- and $\xi_+ \in \bar{I}$ be such that (2.5) holds, for the corresponding eigenfunctions u_0, u_1.

Let $L_V(x) = ax + b$ be the line that intersects the graph of V at ξ_- and $\xi_+ \in \bar{I}$. By the convexity of V,

$$V - L_V \geq 0 \text{ on } (-1, \xi_-), \quad V - L_V \leq 0 \text{ on } (\xi_-, \xi_+), \quad V - L_V \geq 0 \text{ on } (\xi_+, 1).$$

Consequently, using (2.5),

$$(V - L_V)(|u_1|^p - |u_0|^p) \geq 0 \text{ on } \bar{I}.$$

In particular, since V is not affine, there exists a set of positive measure on which the last inequality is strictly positive and hence

$$\int_I (V - L_V)(|u_1|^p - |u_0|^p)\,\mathrm{d}x > 0.$$

For the family $\{V^t\}_{t\in[0,1]}$ given by $V^t = tV + (1-t)L_V$, $\dot{V}^t = V - L_V$ and so (2.1) shows that

$$\frac{\partial}{\partial t}\Gamma_p(V^t) = \int_I \big(V - L_V\big)\left(|u_1|^p - |u_0|^p\right)\mathrm{d}x > 0.$$

Integrating this inequality with respect to t over $(0,1)$ gives

$$\Gamma_p(V) > \Gamma_p(L_V) = \Gamma_p(ax),$$

where ax is the purely linear part of L_V. We can drop the constant term b because adding a constant to the potential shifts all eigenvalues by that constant, and therefore has no effect on the gap. □

5. Further technicalities

In this section we derive some technical results, mostly for linear potentials, which are necessary for proving Theorem 1.4.

Lemma 5.1. *Suppose that $\int x(u_1^2 - u_0^2) = 0$. Then $u_1^2 - u_0^2$ has exactly two interior zeroes, and*

$$u_1(1)^2 - u_0(1)^2 > 0 \text{ and } u_1(-1)^2 - u_0(-1)^2 > 0. \tag{5.1}$$

Proof. From Lemma 2.4, at least one of these is positive, and $u_1^2 - u_0^2$ has either one or two interior zeroes. Suppose that ξ_0 is the *sole* zero of $u_1^2 - u_0^2$, so that $(x - \xi_0)(u_1^2 - u_0^2)$ is nonzero and has the same sign for all $x \neq \xi_0$. Then

$$0 \neq \int (x - \xi_0)(u_1^2 - u_0^2)\,\mathrm{d}x = \int x(u_1^2 - u_0^2)\,\mathrm{d}x - \xi_0 \int u_1^2 - u_0^2\,\mathrm{d}x = 0,$$

where in the last step we use that $\int u_1^2\,\mathrm{d}x = \int u_0^2\,\mathrm{d}x$. The contradiction implies that $u_1^2 - u_0^2$ has two zeros. □

Now we compare the first eigenfunctions in the cases that V is linear and V is zero.

Lemma 5.2. *For $a \geq 0$, let $(u_0^{ax}, \lambda_0^{ax})$ be the first eigenpair of the Schrödinger operator $-\Delta + ax$ with Robin boundary conditions* (1.2). *Then for $a > 0$, the ratio u_0^{ax}/u_0^0 is monotone decreasing along $\bar{I}$.*

Proof. Let $x_1 = (\lambda_0^{ax} - \lambda_0^0)/a$, for $a > 0$. We claim that $x_1 \in I$.

For later convenience, we work with any eigenpair (u^{ax}, λ^{ax}). We can use Lemma 2.1 to write

$$\lambda^{ax} - \lambda^0 = \int_{t=0}^{a} \frac{d}{dr}\lambda^{rx}\, dr = \int_{t=0}^{a}\int_I x\,(u^{rx})^2\, \mathrm{d}x\, dr. \tag{5.2}$$

However, on I, $-(u^{rx})^2 < x\,(u^{rx})^2 < (u^{rx})^2$, and so

$$-1 = -\int_I (u^{rx})^2\, \mathrm{d}x < \int_I x\,(u^{rx})^2\, \mathrm{d}x < \int_I (u^{rx})^2\, \mathrm{d}x = 1.$$

Using this estimate for the integrand of (5.2) leads to

$$-a < \lambda^{ax} - \lambda^0 < a, \tag{5.3}$$

and therefore $x_1 \in (-1, 1)$.

We now write the first eigenfunctions for linear potentials as $u_0^{ax} = \bar{u}_a$ and $u_0^0 = \bar{u}$. For any $x \in (-1, x_1]$,

$$\begin{aligned}
\left(\frac{\bar{u}_a}{\bar{u}}\right)'(x) &= \frac{1}{|\bar{u}|^2}\left(\bar{u}_a{}'\bar{u} - \bar{u}_a\bar{u}'\right)(x)\\
&= \frac{1}{\bar{u}^2}\int_{-1}^{x}\left(\bar{u}_a{}'\bar{u} - \bar{u}_a\bar{u}'\right)'\, dy \text{ where we use the boundary values at } -1\\
&= \frac{1}{\bar{u}^2}\int_{-1}^{x} \bar{u}_a{}''\bar{u} + \bar{u}_a{}'\bar{u}' - \bar{u}_a{}'\bar{u}' - \bar{u}_a\bar{u}''\, dy\\
&= \frac{1}{\bar{u}^2}\int_{-1}^{x}(ay - \lambda_0^{ax} + \lambda_0^0)\bar{u}_a\bar{u}\, dy\\
&< 0,
\end{aligned}$$

where we have used that $ay < ax < ax_1 = \lambda_0^{ax} - \lambda_0^0$. Similarly, for $x \in (x_1, 1)$,

$$\left(\frac{\bar{u}_a}{\bar{u}}\right)'(x) = -\frac{1}{\bar{u}^2}\int_{x}^{1}\left(\bar{u}_a{}'\bar{u} - \bar{u}_a\bar{u}'\right)'\, \mathrm{d}x = -\frac{1}{\bar{u}^2}\int_{x}^{1}(ay - \lambda_0^{ax} + \lambda_0^0)\bar{u}_a\bar{u}\, \mathrm{d}x < 0.$$

□

Lemma 5.3. *For $a \geq 0$, let $(u_0^{ax}, \lambda_0^{ax})$ be the first eigenpair of the Schrödinger operator $-\Delta + ax$ with Robin boundary conditions* (1.2)*. Then for $a > 0$,*

$$|u_0^{ax}|^2(1) - |u_0^{ax}|^2(-1) < 0.$$

Proof. Again we write $u_0^{ax} = \bar{u}_a$ and $u_0^0 = \bar{u}$. From Lemma 5.2, $\bar{u}_a/\bar{u}$ is decreasing, so that for $x > 0$,

$$\frac{\bar{u}_a(-x)}{\bar{u}(-x)} > \frac{\bar{u}_a(x)}{\bar{u}(x)}$$

however $\bar{u}$ is even, so

$$\bar{u}_a{}^2(-x) > \bar{u}_a{}^2(x);$$

then the result follows directly with $x = 1$. □

Lemma 5.4. *For $a > 0$, let λ_1^{ax} be the second eigenvalue of the Schrödinger operator $-\Delta + ax$ for Robin boundary conditions* (1.2) *with parameter α. Then*

$$\alpha^2 + \lambda_1^{ax} + a > 0. \tag{5.4}$$

Proof. From (5.3), we have $\lambda_1^{ax} + a \geq \lambda_1^0$. Then

$$\alpha^2 + \lambda_1^{ax} + a \geq \alpha^2 + \lambda_1^0.$$

We estimate λ_1^0: we have a zero potential and everything may be done explicitly.

If $\alpha \geq 0$ then (5.4) follows directly from the positivity of the Rayleigh quotient.

When $\alpha < 0$, λ_0 is negative, since otherwise the general solution to $-u'' = \lambda u$ is given by $u(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$; however the boundary condition requires that $\sqrt{\lambda}\tan\sqrt{\lambda} = \alpha$, which cannot be satisfied if $\alpha < 0$.

In the case that $-1 < \alpha < 0$, λ_1 is positive: to be precise, $\lambda_1 = \mu^2$, where μ solves $-\alpha\tan\mu = \mu$, and $u_1 = \sin\mu x$. If $\alpha = -1$, then $\lambda_1 = 0$, with a linear eigenfunction. If $\alpha < -1$, then λ_1 is negative, however the claim still holds, since then $\lambda_1 = -\mu^2$ where μ solves $\mu = -\alpha\tanh\mu$, and hence

$$\alpha^2 + \lambda_1 = \alpha^2 - \mu^2 = \left(\frac{\mu}{\tanh\mu}\right)^2 (1 - \tanh\mu^2) > 0.$$

The conclusion follows. □

Lemma 5.5. *Let (u, λ) be an eigenpair of the Schrödinger operator $-\Delta + ax$. Then*

$$\left[(u')^2 + (\lambda - ax)u^2\right]_{-1}^1 = -a \tag{5.5}$$

and

$$\left[(u')^2 + (\lambda - ax + 1)u^2 - 2xuu'\right]_{-1}^1 = 4\lambda\int_{-1}^1 xu^2 - 5a\int_{-1}^1 x^2u^2\,dx. \tag{5.6}$$

Proof. We calculate

$$\begin{aligned}\left[u'^2 + (\lambda - ax)u^2\right]_{-1}^1 &= \int_{-1}^1 \frac{d}{dx}\left[u'^2 + (\lambda - ax)u^2\right] \mathrm{d}x\\ &= \int_{-1}^1 2uu'' + (\lambda - ax)2uu' - au^2\,\mathrm{d}x\\ &= -a\int_{-1}^1 u^2\,\mathrm{d}x = -a.\end{aligned}$$

Next,

$$
\begin{aligned}
&\left[x^2\left(u'^2+(\lambda-ax)u^2\right)-2xuu'+u^2\right]_{-1}^{1}\\
&\qquad\qquad\qquad\qquad = \int_{-1}^{1}\frac{d}{dx}\left[x^2\left(u'^2+(\lambda-ax)u^2\right)-2xuu'+u^2\right]\,\mathrm{d}x\\
&\qquad\qquad\qquad\qquad = \int_{-1}^{1}u^2\left[4x(\lambda-ax)-ax^2\right]\,\mathrm{d}x\\
&\qquad\qquad\qquad\qquad = 4\lambda\int_{-1}^{1}xu^2\,\mathrm{d}x-5a\int_{-1}^{1}x^2u^2\,\mathrm{d}x.
\end{aligned}
$$

□

6. Proof of Theorem 1.3

We begin by estimating the boundary terms of eigenfunctions with linear potentials.

Lemma 6.1. *Let u_0 and u_1 be the first two eigenfunctions of the Schrödinger operator $-\Delta+ax$ with $a>0$, and Robin boundary conditions* (1.2) *with $\alpha\geq-1$. Then*

$$u_1(1)^2-u_0(1)^2-u_1(-1)^2+u_0(-1)^2>0.$$

Proof. We apply (5.5) to u_1 and u_0 in turn:

$$u_1(1)^2\left(\alpha^2+\lambda_1-a\right)-u_1(-1)^2\left(\alpha^2+\lambda_1+a\right)=-a \tag{6.1}$$

$$u_0(1)^2\left(\alpha^2+\lambda_0-a\right)-u_0(-1)^2\left(\alpha^2+\lambda_0+a\right)=-a. \tag{6.2}$$

Adding $2au_1(1)^2$ to both sides of (6.1) allows it to be rearranged as

$$
\begin{aligned}
\left(\alpha^2+\lambda_1+a\right)\left(u_1(1)^2-u_1(-1)^2\right)&=-a+2au_1(1)^2\\
&>-a+2au_0(1)^2
\end{aligned}
$$

where in the last line we have used that $u_1(1)^2-u_0^2(1)>0$, as in (5.1). Similarly we can use (6.2) to find

$$
\begin{aligned}
-a+2au_0(1)^2&=\left(\alpha^2+\lambda_0+a\right)\left(u_0(1)^2-u_0(-1)^2\right)\\
&>\left(\alpha^2+\lambda_1+a\right)\left(u_0(1)^2-u_0(-1)^2\right)
\end{aligned}
$$

where we have used that $u_0(1)^2-u_0(-1)^2<0$, by Lemma 5.3, and $\lambda_0<\lambda_1$.

Combining both these calculations we have

$$\left(\alpha^2+\lambda_1+a\right)\left(u_1(1)^2-u_1(-1)^2\right)>\left(\alpha^2+\lambda_1+a\right)\left(u_0(1)^2-u_0(-1)^2\right),$$

and since $(\alpha^2+\lambda_1+a)>0$ by Lemma 5.4, the result follows. □

In order to prove Theorem 1.3, we need to show that the gap for linear potentials ax achieves its minimum for some finite a; later, we'll show that in fact this occurs at $a = 0$.

Lemma 6.2. *Let $\Gamma_2(ax)$ be the fundamental gap for the Schrödinger operator $-\Delta + ax$, with Robin boundary conditions* (1.2)*. Then $\Gamma_2(ax) \to \infty$ as $|a| \to \infty$.*

Proof. We only consider the case $a > 0$ since the case $a < 0$ proceeds similarly. Let (u_i, λ_i^a) be an eigenpair for the given operator. We rescale the domain using $s = (x+1)a^{-1/3}$, so that $w_i(s) := u_i(-1 + a^{-1/3}s)$ solves

$$\begin{cases} -w_i'' + sw_i = \hat{\lambda}_i^a w_i & \text{on } (0, 2a^{1/3}) \\ w_i'(0) - \alpha a^{-1/3} w_i(0) = 0, & w_i'(2a^{1/3}) + \alpha a^{-1/3} w_i(2a^{1/3}) = 0, \end{cases}$$

where

$$\hat{\lambda}_i^a = a^{1/3} + a^{-2/3}\lambda_i^a. \tag{6.3}$$

As $a \to \infty$, the eigenpair $(w_i, \hat{\lambda}_i^a)$ approaches the pair (v_i, μ_i): in the case that $\alpha \in \mathbb{R}$, v_i is the $L^2(0, +\infty)$-solution to

$$-v_i'' + sv_i = \mu_i v_i \text{ on } (0, \infty) \tag{6.4}$$

with $\mu_i = \mu_i^N > 0$ and Neumann boundary condition $v_i'(0) = 0$; or if $\alpha = \infty$, v_i is the solution to (6.4) with $\mu_i = \mu_i^D > 0$ and Dirichlet boundary condition $v_i(0) = 0$.

In both cases the function v_i and the eigenvalues μ_i^N and μ_i^D are explicitly known: in the first case, $v_i(s) = \mathrm{Ai}(s - \mu_i^N)$, where Ai is the *Airy function*, the bounded solution to the ODE $y''(x) = xy(x)$, and the shift μ_i^N is such that $\mathrm{Ai}'(-\mu_i^N) = 0$. For $i = 0$, μ_0^N is such that $v_0 > 0$ on $(0, \infty)$, and for $i = 1$, μ_1^N is such that v_1 changes sign once on $(0, \infty)$. The second case is similar with $v_i(s) = \mathrm{Ai}(s - \mu_i^D)$, where $\mathrm{Ai}(-\mu_i^D) = 0$.

In either case, $\hat{\lambda}_i^a \to \mu_i$ as $a \to +\infty$ and so, by (6.3),

$$\lambda_i^a = a^{2/3}\mu_i - a + \mathcal{O}(a^{2/3}) \qquad \text{as } a \to +\infty.$$

Applying this expansion of λ_i^a to the fundamental gap $\Gamma_2(ax)$ yields

$$\Gamma_2(ax) = a^{2/3}(\mu_2 - \mu_1) + \mathcal{O}(a^{2/3}) \qquad \text{as } a \to +\infty.$$

As $\mu_2 - \mu_1 > 0$, we find $\Gamma_2(ax) \to \infty$ as $a \to \infty$. □

Proof of Theorem 1.3. Due to Lemma 6.2, the gap for linear potentials $\Gamma(ax)$ achieves a minimum at some $a \in \mathbb{R}$. Define a family of potentials $V^t := tax$. Using (2.1),

$$\frac{d}{dt}(\lambda_1^{V^t} - \lambda_0^{V^t}) = a \int x(u_1^2 - u_0^2)\,\mathrm{d}x,$$

where here and in the remainder of this section u_0 and u_1 are eigenfunctions for the problem (1.1)–(1.2) with linear potential $V^t = tax$.

Suppose, in order to obtain a contradiction, that the critical point of the gap occurs at some $a > 0$. This implies that

$$\int x(u_1^2 - u_0^2)\,\mathrm{d}x = 0. \tag{6.5}$$

Next, under Robin or Neumann boundary conditions, identity (5.5) becomes

$$(\lambda - a + \alpha^2)u(1)^2 - (\lambda + a + \alpha^2)u(-1)^2 = -a; \tag{6.6}$$

and identity (5.6) becomes

$$u(1)^2\left[\alpha^2 + \lambda - a + 1 + 2\alpha\right] - u(-1)^2\left[\alpha^2 + \lambda + a + 1 + 2\alpha\right] = 4\lambda\int_{-1}^{1} xu^2\,\mathrm{d}x - 5a\int_{-1}^{1} x^2u^2\,\mathrm{d}x,$$

subtracting (6.6) from this results in

$$u(1)^2\left[1 + 2\alpha\right] - u(-1)^2\left[1 + 2\alpha\right] - a = 4\lambda\int_{-1}^{1} xu^2\,\mathrm{d}x - 5a\int_{-1}^{1} x^2u^2\,\mathrm{d}x.$$

Applying this to both u_0 and u_1, and subtracting, we find

$$\begin{aligned}\left[1 + 2\alpha\right]&\left[u_1(1)^2 - u_0(1)^2 - u_1(-1)^2 + u_0(-1)^2\right] \\ &= 4(\lambda_1 - \lambda_0)\int_{-1}^{1} x(u_1^2 - u_0^2)\,\mathrm{d}x - 5a\int_{-1}^{1} x^2(u_1^2 - u_0^2)\,\mathrm{d}x.\end{aligned} \tag{6.7}$$

The assumption $\alpha \geq -\frac{1}{2}$, and Lemma 6.1, imply that the left hand side is non-negative. However, we claim that the right hand side is strictly negative.

The first term of the right hand side is zero, by our assumption (6.5). The final term of (6.7) can be estimated by the same trick we used in Theorem 1.2: let $cx + b$ be the line that intersects x^2 at ξ_- and ξ_+, which are the points where $u_1^2 - u_0^2$ changes sign, as in (2.5). Then $(x^2 - cx - b)(u_1^2 - u_0^2)$ is strictly positive for all $x \neq \xi_\pm$. Furthermore, $\int u_1^2 = \int u_0^2$ and our assumption (6.5) is that $\int x(u_1^2 - u_0^2) = 0$, hence

$$-5a\int_{-1}^{1} x^2(u_1^2 - u_0^2)\,\mathrm{d}x = -5a\int_{-1}^{1}(x^2 - cx - b)(u_1^2 - u_0^2)\,\mathrm{d}x < 0.$$

With these two observations, (6.7) becomes

$$\left[1 + 2\alpha\right]\left[u_1(1)^2 - u_0(1)^2 - u_1(-1)^2 + u_0(-1)^2\right] < 0,$$

as claimed. The contradiction implies our original assumption (6.5) must be false, and thus the minimum of the gap is not attained for any potential ax with $a \neq 0$. Thus the zero potential, with $a = 0$, minimises the gap over all linear potentials. □

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