PROOFS OF URYSOHN'S LEMMA AND THE TIETZE EXTENSION THEOREM VIA THE CANTOR FUNCTION

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Abstract. Urysohn's Lemma is a crucial property of normal spaces that deals with separation of closed sets by continuous functions. It is also a fundamental ingredient in proving the Tietze Extension Theorem, another property of normal spaces that deals with the existence of extensions of continuous functions. Using the Cantor function, we give alternative proofs for Urysohn's Lemma and the Tietze Extension Theorem.

1. Introduction

Urysohn's Lemma provides the means for proving big theorems in topology such as Urysohn's metrization theorem (see Urysohn's final[1] paper [17]) and the Tietze Extension Theorem proved by Tietze [15] for metric spaces and generalised by Urysohn [16] to normal spaces. For further details, see [6]. Using the Cantor function, we give new proofs for Urysohn's Lemma (in Section 2) and the Tietze Extension Theorem (in Section 3). Urysohn's Lemma, the origin of which is in the third appendix to Urysohn's paper [16], gives a property that characterises normal spaces[2]:

Theorem 1.1 (Urysohn's Lemma)**.** If A and B are disjoint closed subsets of a normal space X, then there exists a continuous function $f : X \to [0, 1]$ such that $f = 0$ on A and $f = 1$ on B.

Munkres, the author of the popular book [11], regards the Urysohn Lemma as "the first deep theorem of the book" (see p. 207 in [11]). He adds that "it would take considerably more originality than most of us possess to prove this lemma unless we were given copious hints." For the standard proof of Urysohn's Lemma, see [8, p. 115], [11, p. 207] or [19, p. 102].

A function as in Theorem 1.1 is called a *Urysohn function*. Its existence is crucial to any of the many approaches to the Tietze Extension Theorem ([1, 7, 10, 12, 13]). But, surprisingly, apart from the classical one, it seems that no other constructions of a Urysohn function are known.

We reduce the proof of Theorem 1.1 to the case of a connected normal space when we construct a new Urysohn function. Our argument neither relies on nor reduces to the standard proof (see Remark 2.1). We use the Cantor set and Cantor function, which we introduce next.

Algebraically, a point $p \in [0, 1]$ is in the Cantor set $\mathcal{C}$ if and only if p has a ternary expansion that does not use digit 1. We thus write $p = 0.p_1p_2 \ldots p_k \ldots_3$, where $p_k \in \{0, 2\}$ for every $k \geq 1$. (The subscript 3 indicates that the expansion is in base 3.) Geometrically, we construct the Cantor set as follows. Starting with the interval $[0, 1]$, we remove its open middle third interval $(1/3, 2/3)$. We apply the same process to the remaining intervals $[0, 1/3]$ and $[2/3, 1]$. The Cantor set $\mathcal{C}$ is the set of points in $[0, 1]$ that remain after continuing this removal process *ad infinitum*. At each stage $n \geq 1$ in this construction, we remove 2^{n-1} open intervals $(\alpha^{(n)}, \beta^{(n)})$, where

$$\alpha^{(n)} = 0.\alpha_1^{(n)} \ldots \alpha_{n-1}^{(n)} 1_3, \qquad \beta^{(n)} = 0.\alpha_1^{(n)} \ldots \alpha_{n-1}^{(n)} 2_3. \tag{1.1}$$

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[1]At the age of only 26, he drowned while swimming in the ocean.

[2]A topological space X is *normal* if every disjoint closed subsets of X can be included in disjoint open sets.

Here, $\alpha_k^{(n)} \in \{0,2\}$ for each $1 \le k \le n-1$. Let $\mathcal{L} = \cup_{n=1}^{\infty} \mathcal{L}_n$, where $\mathcal{L}_n$ is the collection of all points $\alpha^{(n)}$ with a ternary representation as in (1.1). Hence, $\mathcal{L}_1 = \{1/3\}$ and $\mathcal{L}_2 = \{1/9, 7/9\}$. Now, $\mathcal{L}$ is countable and dense in $\mathcal{C}$ (but not dense in $[0,1]$). All the numbers in $\mathcal{L}$ are "endpoints" and right limit points of $\mathcal{C}$. (Each number in $\mathcal{L}$ has a ternary expansion consisting entirely of 0s and 2s. For example, $1 = 0.222\ldots_3$, $1/3 = 0.0222\ldots_3$ and $7/9 = 0.20222\ldots_3$.)

The Cantor function $\Phi : [0,1] \to [0,1]$ is continuous, non-decreasing and surjective. It is given by $\Phi = \sum_{k=1}^{n-1} \alpha_k^{(n)} 2^{-k-1} + 2^{-n}$ on $\left[\alpha^{(n)}, \beta^{(n)}\right]$ and $\Phi(p) = \sum_{k=1}^{\infty} p_k 2^{-(k+1)}$ for every $p \in \mathcal{C}$, where $p = 0.p_1p_2\ldots p_k\ldots_3$ with $p_k \in \{0,2\}$ for every $k \ge 1$. The binary expansion of any $y \in [0,1]$ can be translated into a ternary representation of a number $p \in \mathcal{C}$ by replacing all the 1s by 2s. Hence, $\Phi(\mathcal{C}) = [0,1]$ and $\Phi(\mathcal{L}) = \mathcal{D} \setminus \{0,1\}$, where $\mathcal{D}$ is the set of all dyadic rationals in $[0,1]$.

Notation. Fix $\alpha^{(n)} \in \mathcal{L}_n$ and $\beta^{(n)} = \alpha^{(n)} + 3^{-n}$. For $k \ge 1$, let $q_k^{(n)} := \alpha^{(n)} - 2 \cdot 3^{-n-k}$ and $\ell_k^{(n)} := \beta^{(n)} + 3^{-n-k}$. Then, $\{q_k^{(n)}\}_k$ and $\{\ell_k^{(n)}\}_k$ are strictly monotone sequences in $\mathcal{L}_{n+k}$ converging to $\alpha^{(n)}$ and $\beta^{(n)}$, respectively, satisfying $\Phi(q_k^{(n)}) \nearrow \Phi(\alpha^{(n)})$ and $\Phi(\ell_k^{(n)}) \searrow \Phi(\alpha^{(n)})$ as $k \to \infty$. If $p_* = \max\{p \in \cup_{j=1}^n \mathcal{L}_j \cup \{0\} :\ p < \alpha^{(n)}\}$ and $p^* = \min\{p \in \cup_{j=1}^n \mathcal{L}_j \cup \{1\} :\ p > \beta^{(n)}\}$, then $p_*, \alpha^{(n)}$ and p^* are *consecutive* points in $\cup_{j=1}^n \mathcal{L}_j \cup \{0,1\}$.

2. Proof of Urysohn's Lemma

The connected components of any topological space X form a partition of X and each connected component of X is closed. Since normality is closed-hereditary, it is sufficient to prove Urysohn's Lemma when X is a connected normal space. Suppose A and B are disjoint closed subsets of such a space X. Set $U_0 := A$ and $U_1 := X \setminus B$.

Step 1. We inductively generate a family $\{U_p\}_{p \in \mathcal{L}}$ of open neighbourhoods of A such that $\overline{U_p} \subset U_q$ for all $p, q \in \cup_{n \ge 1} \mathcal{L}_n \cup \{1\}$ with $p < q$. For $n = 1$, by the normality of X, the set U_1 contains (strictly) the closure of an open neighbourhood $U_{1/3}$ of A.

Fix $n \ge 1$. Assume that $\{U_p\}_{p \in \cup_{j=1}^n \mathcal{L}_j}$ is a family of open neighbourhoods of A satisfying

$$\overline{U_p} \subset U_q \quad \text{for all } p, q \in \cup_{j=1}^n \mathcal{L}_j \cup \{1\} \text{ with } p < q. \tag{$\mathfrak{B}_n$}$$

Let $\alpha^{(n)} \in \mathcal{L}_n$ be arbitrary. Then, $q_1^{(n)} \in (p_*, \alpha^{(n)})$ and $\ell_1^{(n)} \in (\alpha^{(n)}, p^*)$ are consecutive points in $\mathcal{L}_{n+1}$. By the induction assumption, U_{p^*} and $U_{\alpha^{(n)}}$ are open neighbourhoods of $\overline{U_{\alpha^{(n)}}}$ and $\overline{U_{p_*}}$, respectively. Thus, U_{p^*} contains the closure of an open neighbourhood $U_{\ell_1^{(n)}}$ of $\overline{U_{\alpha^{(n)}}}$, whereas $U_{\alpha^{(n)}}$ contains the closure of an open neighbourhood $U_{q_1^{(n)}}$ of $\overline{U_{p_*}}$. The collection of all $U_{q_1^{(n)}}$ and $U_{\ell_1^{(n)}}$ obtained by varying $\alpha^{(n)} \in \mathcal{L}_n$ yields the family $\{U_q\}_{q \in \mathcal{L}_{n+1}}$ of open sets satisfying $(\mathfrak{B}_{n+1})$.

Step 2. We define $g = 1$ on $X \setminus U_1$, $g = 0$ on A and $g(x) = \inf\{p \in \mathcal{L} :\ x \in U_p\}$ for every $x \in U_1 \setminus A$. If $g(x) > p$ for $p \in \mathcal{L}$, then $x \notin \overline{U_p}$. Otherwise, $x \in U_q$ for every $q \in \mathcal{L}$ with $q > p$. Then, $g(x) \le q$. By letting $q \in \mathcal{L}$ with $q \searrow p$, we arrive at $g(x) \le p$, which is a contradiction.

Let $F = \Phi \circ g$ on X. Then, $F = 0$ on A and $F = 1$ on B. We prove that $F : X \to [0,1]$ is continuous. For any $\zeta \in \mathcal{D} \setminus \{0,1\}$, there exist $n \ge 1$ and $\alpha^{(n)} \in \mathcal{L}_n$ such that $\zeta = \Phi(\alpha^{(n)})$.

We have $F^{-1}([0,\zeta)) = \cup_{\xi \in \mathcal{L} \cap (0, \alpha^{(n)})} U_\xi$. Indeed, if $x \in U_\xi$ for $\xi \in \mathcal{L} \cap (0, \alpha^{(n)})$, then $g(x) \le \xi$, which gives that $F(x) = \Phi(g(x)) \le \Phi(\xi) < \Phi(\alpha^{(n)}) = \zeta$. Conversely, if $x \in F^{-1}([0,\zeta))$, then $F(x) < \Phi(q_k^{(n)}) < \zeta$ by taking $k \ge 1$ large enough. Hence, $g(x) < q_k^{(n)}$ so that $x \in U_{q_k^{(n)}}$.

Similarly, we see that $F^{-1}((\zeta,1]) = \cup_{\eta \in \mathcal{L} \cap (\beta^{(n)}, 1)} (X \setminus \overline{U_\eta})$. Indeed, let $x \in X \setminus \overline{U_\eta}$ for $\eta \in \mathcal{L}$ with $\eta > \beta^{(n)}$. Then, $g(x) \ge \eta$ and, hence, $F(x) \ge \Phi(\eta) > \zeta$. Conversely, if $x \in F^{-1}((\zeta,1])$, then $F(x) > \Phi(\ell_k^{(n)}) > \zeta$ for $k \ge 1$ large enough. We have $g(x) > \ell_k^{(n)}$ and, hence, $x \notin \overline{U_{\ell_k^{(n)}}}$.

As $\mathcal{S} = \{[0,\tau),\ (\tau,1] :\ 0 < \tau < 1\}$ is a subbase for $[0,1]$ and $\mathcal{D}$ is dense in $[0,1]$, the continuity of F follows using that $F^{-1}([0,\zeta))$ and $F^{-1}((\zeta,1])$ are open for any dyadic rational ζ in $(0,1)$. $\square$

Remark 2.1. The standard approach of Urysohn's Lemma comprises three steps: (i) construction of a family $\{U_r\}_{r\in\mathcal{D}}$ of open sets indexed by[3] the dyadic rationals $r = j/2^k$ in the interval $[0,1]$ such that $A \subseteq U_0$, $B = X \setminus U_1$ and $\overline{U_r} \subseteq U_s$ whenever $r < s$; (ii) verification by induction that the family of open sets $\{U_r\}_{r\in\mathcal{D}}$ has the required properties; (iii) construction of the continuous function: $f(x) = \inf\{r : x \in U_r\}$ for $x \in X \setminus B$ and $f = 1$ on B.

We observe that for our family $\{U_p\}_{p\in\mathcal{L}}$ of open sets, the index set $\mathcal{L}$ is *not* dense in $[0,1]$. It is dense in the Cantor set. Moreover, the continuity of our Urysohn function $F(= \Phi \circ g)$ follows essentially as a result of composing the Cantor function with g. And g is *never* continuous on the connected space X since g takes values in the Cantor set (a perfect set that is nowhere dense).

3. Proof of the Tietze Extension Theorem

Using our new Urysohn function, we give an alternative proof of the Tietze Extension Theorem (see Theorem 3.1). We use the following result, which is easy to establish (see [12, Lemma 1]).

Lemma 1. *Let E and Y be closed subspaces in a normal space X and let U be an open neighbourhood of Y in X. Let a subset C of E be a closed neighbourhood in E of $Y \cap E$ such that $C \subseteq U \cap E$. Then, Y admits a closed neighbourhood Z that is included in U and $Z \cap E = C$.*

Theorem 3.1. Let E be a closed subspace of a normal space X. Then, every continuous function $f : E \to [0,1]$ can be extended to a continuous function $F : X \to [0,1]$.

Proof. As for Urysohn's Lemma, we can assume that X is a connected normal space.
Case I. Let $f : E \to [0,1]$ be a continuous and *surjective* function. The sets $A = f^{-1}(0)$ and $B = f^{-1}(1)$ are disjoint and closed in E (and, hence, in X). Define $U_0 = A$ and $U_1 = X \setminus B$.

For $Z \subseteq X$, we set $Z^c := X \setminus Z$. We construct open neighbourhoods $\{U_p\}_{p\in\mathcal{L}}$ of A as in Step 1 of Urysohn's Lemma and, in addition, $U_p \cap E = f^{-1}([0, \Phi(p)))$ for every $p \in \mathcal{L}$. More precisely, for each $n \geq 1$, we generate open neighbourhoods $\{U_q\}_{q\in\mathcal{L}_n}$ of A satisfying $(\mathfrak{B}_n)$ and

$$U_q^c \cap E = f^{-1}([\Phi(q), 1]) \quad \text{for every } q \in \mathcal{L}_n. \qquad (\mathfrak{D}_n)$$

By Lemma 1, B has a closed neighbourhood $U_{1/3}^c$ contained in A^c with $U_{1/3}^c \cap E = f^{-1}([1/2, 1])$. This proves the claim for $n = 1$. For $n \geq 1$, assume that $\{U_p\}_{p\in\cup_{j=1}^n \mathcal{L}_j}$ is a family of open neighbourhoods of A satisfying $(\mathfrak{B}_n)$ and $(\mathfrak{D}_n)$. For fixed $\alpha^{(n)} \in \mathcal{L}_n$, let $p_*, p^*, q_1^{(n)}$ and $\ell_1^{(n)}$ be as in §2. Using the induction assumption and Lemma 1, we find that $U_{p^*}^c$ has a closed neighbourhood $U_{\ell_1^{(n)}}^c$ contained in $(\overline{U_{\alpha^{(n)}}})^c$ and $(\mathfrak{D}_{n+1})$ holds for $q = \ell_1^{(n)} \in \mathcal{L}_{n+1}$. Similarly, $U_{\alpha^{(n)}}^c$ has a closed neighbourhood $U_{q_1^{(n)}}^c$ contained in $(\overline{U_{p_*}})^c$ and $(\mathfrak{D}_{n+1})$ holds for $q = q_1^{(n)} \in \mathcal{L}_{n+1}$. All $U_{q_1^{(n)}}$ and $U_{\ell_1^{(n)}}$ obtained by varying $\alpha^{(n)} \in \mathcal{L}_n$ yield the family $\{U_q\}_{q\in\mathcal{L}_{n+1}}$ satisfying $(\mathfrak{B}_{n+1})$ and $(\mathfrak{D}_{n+1})$.

Let $F : X \to [0,1]$ be our Urysohn function associated to $\{U_p\}_{p\in\mathcal{L}}$. For any $\zeta \in \mathcal{D} \setminus \{0,1\}$, there exist $n \geq 1$ and $\alpha^{(n)} \in \mathcal{L}_n$ with $\zeta = \Phi(\alpha^{(n)}) = \Phi(\beta^{(n)})$. By the density of $\mathcal{L}$ in $\mathcal{C}$, for every $\rho \in \mathcal{L}$ with $\rho > \beta^{(n)}$, there exists $\eta \in \mathcal{L} \cap (\beta^{(n)}, \rho)$, which yields that $U_\rho^c \subseteq X \setminus \overline{U_\eta}$. Then, by Step 2 in §2, $F^{-1}((\zeta, 1]) = \cup\{U_\rho^c : \rho \in \mathcal{L},\ \rho > \beta^{(n)}\}$ and $F^{-1}([0,\zeta)) = \cup\{U_\xi : \xi \in \mathcal{L},\ \xi < \alpha^{(n)}\}$. Thus, $E \cap F^{-1}((\zeta, 1]) = f^{-1}((\zeta, 1])$ and $E \cap F^{-1}([0, \zeta)) = f^{-1}([0, \zeta))$. These equalities extend to every $\zeta \in (0,1)$ by density of $\mathcal{D}$ in $[0,1]$. Hence, $F : X \to [0,1]$ is a continuous extension of f.
Case II. Let $h : E \to [0,1]$ be any continuous function ($E \subset X$ is closed). We choose open sets V_1 and V_2 such that $E \subset V_1$ and $\overline{V_1} \subset V_2$. Urysohn's Lemma gives a continuous function $\varphi : X \to [0,1]$ with $\varphi = 0$ on $\overline{V_1}$ and $\varphi = 1$ on V_2^c. We have $\varphi(\overline{V_2} \setminus V_1) = [0,1]$ by the connectedness of X. If $f = \varphi$ on $\overline{V_2} \setminus V_1$ and $f = h$ on E, then $f : (\overline{V_2} \setminus V_1) \cup E \to [0,1]$ is continuous and surjective. By Case I, f (and thus h) has a continuous extension $F : X \to [0,1]$. $\square$

[3] The index set $\mathcal{D}$ can be any subset of $\mathbb{Q}$ that is dense in $[0,1]$.

4. Notes on the Cantor set and Cantor function

The Cantor set[4] and Cantor function are two of Cantor's ingenious creations that go back to 1883. During the years 1879–1884, G. Cantor (1845–1918) gave the first systematic treatment of the point set topology of the real line in a series of papers entitled "Über unendliche, lineare Punktmannichfaltigkeiten." Among the terms he introduced and still in current use, we mention two: *everywhere dense* set and *perfect set*. The terminology (but not the concept[5]) of "limit point", along with the notion of derived set, was introduced by Cantor in a paper of 1872.

The Cantor set ranks as one of the most frequently quoted fractal objects in the literature. It emerges again and again in many areas of mathematics from topology, analysis and abstract algebra to fractal geometry [9, 18]. The Cantor set appeared in a footnote to Cantor's statement [2] that perfect sets need not be everywhere dense. Without any indication on how he came upon it, Cantor noted that this set is an infinite and perfect set that is *nowhere dense*[6] in any interval, regardless of how small it is. The first occurrence of the Cantor function is in a letter by Cantor [3] dated November 1883. The Cantor function in [3] served as a counterexample to Harnack's extension of the Fundamental Theorem of Calculus to discontinuous functions.

The properties of the Cantor function (also called the Lebesgue function or the Devil's Staircase) are surveyed in [4]. For the history of the Cantor set and Cantor function, see [5].

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[4] Fleron [5] noted that Cantor was not the first to uncover general "Cantor sets." Such sets featured earlier in a paper [14] of Smith, who discovered and constructed nowhere dense sets with outer content.

[5] The concept of limit point was conceived by Weierstrass who, without giving it a name, used it between 1865 and 1886 as part of his statement that *any infinite bounded set in n-dimensional Euclidean space has a limit point* (the Bolzano–Weierstrass Theorem).

[6] A set is called nowhere dense if the interior of its closure is empty.