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More insights into the Trudinger-Moser inequality with monomial weight

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Abstract In this paper we present a detailed study of critical embeddings of weighted Sobolev spaces into weighted Orlicz spaces of exponential type for weights of monomial type. More precisely, we give an alternative proof of a recent result by N. Lam [NoDEA 24(4), 2017] showing the optimality of the constant in the Trudinger-Moser inequality. We prove a Poincaré inequality for this class of weights. We show that the critical embedding is optimal within the class of Orlicz target spaces. Moreover, we prove that it is not compact, and derive a corresponding version of P.-L. Lions' principle of concentrated compactness.

Keywords Trudinger-Moser inequality · monomial weight · compact embedding · concentrated compactness

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1 Introduction

Embedding theorems are known to be very important in the theory of PDE's. Let us recall some classical ones. Assuming that Ω is an open subset of $\mathbb{R}^n$, $n \geq 2$, of finite Lebesgue measure and $W_0^{1,p}(\Omega)$, $p \geq 1$, the *Sobolev space* obtained as the closure of $C_0^\infty(\Omega)$ (the space of infinitely differentiable functions compactly supported in Ω) with respect to the norm $\|\nabla u\|_{L^p(\Omega)}$ (the usual L^p-norm of the Euclidean length of the gradient of u) then the

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following *embeddings* are available

$$W_0^{1,p}(\Omega) \hookrightarrow L^{\frac{np}{n-p}}(\Omega) \quad \text{if } p \in [1,n), \tag{1.1}$$

$$W_0^{1,p}(\Omega) \hookrightarrow L^{\infty}(\Omega) \quad \text{if } p > n. \tag{1.2}$$

(The symbol $X \hookrightarrow Y$ means that the space X is embedded into the space Y, that is, $X \subset Y$ and $\|u\|_Y \le c\|u\|_X$ for all $u \in X$ with a positive constant c independent of u. As usual, symbols $\|u\|_X$, $\|u\|_Y$ denote norms of the function u in the spaces X, Y, respectively.)

In the case when $p = n$, which is usually called *critical*, the situation is qualitatively different from the previous ones. On the one hand, it is known that the space $W_0^{1,n}(\Omega)$ is embedded into $L^q(\Omega)$ for every $q \in [1,\infty)$, however it is not embedded into $L^\infty(\Omega)$. On the other hand, one can immediately see that, for any $q \in [1,\infty)$, $L^q(\Omega)$ is not the optimal target space within the class of Lebesgue spaces. This is in contrast to the subcritical embedding (1.1), where $L^{\frac{np}{n-p}}(\Omega)$ is the optimal target space in this class. It is due to the pioneering work by Trudinger [25] that

$$W_0^{1,n}(\Omega) \hookrightarrow L^{\Phi}(\Omega), \tag{1.3}$$

where $L^{\Phi}(\Omega)$ is the Orlicz space with the Young function $\Phi(t) = \exp(t^{n/(n-1)}) - 1$ (note that such an embedding has been announced earlier by Yudovich [26] without a proof and, in a slightly weaker form, proved by Pokhozhaev [21]). One of the natural questions, is whether the embedding (1.3) is optimal (in some sense). In fact, it can be proved (and it also follows from a more general result which we prove here) that the embedding (1.3) is optimal in the sense that there is no smaller Orlicz space $L^{\Phi_s}(\Omega)$ (that is $L^{\Phi_s}(\Omega) \subsetneqq L^{\Phi}(\Omega)$) such that $W_0^{1,n}(\Omega) \hookrightarrow L^{\Phi_s}(\Omega)$ holds. It might be of interest that the optimality of (1.3) (unlike (1.1)) is not true within a larger class of target spaces. Namely, there exists a Lorentz space $X(\Omega)$ such that $W_0^{1,n}(\Omega) \hookrightarrow X(\Omega)$ and $X(\Omega) \subsetneqq L^{\Phi}(\Omega)$. More details about that can be found in [8] or in the survey paper [19].

Other interesting questions concern the compactness of Sobolev embedings. This property is usually used for proving the existence of an appropriate weak solution of a certain PDE. It is known that similarly to embedding (1.1), embedding (1.3) is not compact. Indeed, it was shown by Hempel, Morris and Trudinger in [11]. Nevertheless, P. L. Lions derived the so called "principle of concentrated compactness" (see [15]) which serves as a powerful tool, e.g., for proving the existence of a weak nontrivial solution to the Dirichlet problem for the quasi-linear n-Laplace operator with nonlinearities in the critical growth range. The first step to obtain Lions' principle is the following result by Moser [16]

$$\sup_u \int_\Omega \exp\left((\alpha|u(x)|)^{n/(n-1)}\right) \mathrm{d}x \quad \begin{cases} < \infty, \text{ if } \alpha \le n s_n^{1/n}, \\ = \infty, \text{ if } \alpha > n s_n^{1/n}, \end{cases} \tag{1.4}$$

where the supremum is taken over all $u \in W_0^{1,n}(\Omega)$ satisfying $\|\nabla u\|_{L^n(\Omega)} \le 1$ (s_n stands for the volume of the unit ball in $\mathbb{R}^n$).

Recently, Cabré and Ros-Oton [4] established the following weighted version of embedding (1.3), that is, the embedding

$$W_0^{1,D}(\Omega,\mu) \hookrightarrow L^{\Phi}(\Omega,\mu), \tag{1.5}$$

where $\Phi(t) = \exp(t^{D/(D-1)}) - 1$, $t \ge 0$, and the measure $\mu = m_A$ is generated by the *monomial weight*, that is,

$$\mathrm{d}m_A(x) = x^A\,\mathrm{d}x = |x_1|^{A_1} \cdots |x_n|^{A_n}\,\mathrm{d}x, \tag{1.6}$$

where

$$x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \quad A = (A_1, \ldots, A_n) \text{ with } A_1 \geq 0, \ldots, A_n \geq 0 \tag{1.7}$$

and

$$D = n + A_1 + \cdots + A_n.$$

Their proof is analogical to Trudinger's one, which is based on the Taylor expansion of the exponential function. Later, Lam [12] improved this result in [4] to Moser's result (1.4) for the measure $\mu = m_A$. He found a constant $\alpha_{D,A} > 0$ such that

$$S_{\alpha,\mu}(\Omega) = \sup_u \int_\Omega \exp\left((\alpha|u(x)|)^{D/(D-1)}\right) \mathrm{d}\mu(x)$$

is finite for every $\alpha \leq \alpha_{D,A}$, and $S_{\alpha,\mu}(\Omega) = \infty$ if $\alpha > \alpha_{D,A}$. Note, in $S_{\alpha,\mu}(\Omega)$ the supremum is taken over all functions u from to the unit ball of the weighted Sobolev space $W_0^{1,D}(\Omega, m_A)$.

It is not difficult to see that embedding (1.5) for $\mu = m_A$ follows from the fact that $S_{\alpha,\mu}(\Omega)$ is finite for every $\alpha \leq \alpha_{D,A}$ (cf. Corollary 2.9 and its proof in Section 3).

In this paper, we

(i) present (in the proof of Theorem 2.7) an alternative approach to Lam's result that $S_{\alpha,m_A}(\Omega) < \infty$ if $0 < \alpha \leq \alpha_{D,A}$;
(ii) establish (in Theorem 2.8) a Poincaré inequality for functions of the Sobolev space $W_0^{1,D}(\Omega, m_A)$;
(iii) deduce (in Corollary 2.9) from Theorem 2.7 embedding (1.5);
(iv) prove (in Theorem 2.10) that, for embedding (1.5), the target space $L^\Phi(\Omega, m_A)$ is optimal within the class of Orlicz spaces;
(v) establish (in Theorem 2.12) that embedding (1.5) is not compact;
(vi) show (in Theorem 2.10) that the Sobolev space $W_0^{1,D}(\Omega, m_A)$ is compactly embedded to any Orlicz space $L^\Psi(\Omega, m_A)$ larger than $L^\Phi(\Omega, m_A)$, that is, to any Orlicz space $L^\Psi(\Omega, m_A)$ such that $L^\Phi(\Omega, m_A)$ is properly embedded to $L^\Psi(\Omega, m_A)$;
(vii) derive (in Theorem 2.13) a corresponding principle of concentrated compactness.

It is easy to see that if $A_1 = \cdots = A_n = 0$ then $\mathrm{d}m_A(x) = \mathrm{d}x$, $p = D = n$, $q = n/(n-1)$ and we obtain the unweighted case (1.3). We would like to emphasize that for proving (iv)–(vii) it is reasonable to assume that $0 \in \Omega$. Otherwise the measure (1.6) might be equivalent to the Lebesgue measure, which results that $W_0^{1,D}(\Omega, m_A) = W_0^{1,D}(\Omega)$ and $L^\Phi(\Omega, m_A) = L^\Phi(\Omega)$. Then, if some of the numbers A_i, $i \in \{1, \ldots, n\}$, is positive, then $D > n$, which means, that we have the embedding (1.2) and the assertions (iv)–(vii) are meaningless.

A well-known approach for proving that of $S_{\alpha,m_A}(\Omega) < \infty$ if $0 < \alpha \leq \alpha_{D,A}$ is based on symmetrization. It enables to convert the n-dimensional case to a one-dimensional one. In particular, Lam [12] used a symmetrization based on results of Talenti and of Cabré and Ros-Oton (see Lemma 5.9 (iii) below). Our approach is a little different and in a sense easier to understand. We express a smooth and compactly supported function u (recall that these functions form a dense subset of Sobolev space $W_0^{1,D}(\Omega, m_A)$) as a certain convolution operator of its gradient. To obtain the main inequality we use a suitable version of an inequality of O'Neil [18] giving an upper estimate of a rearrangement of function u by an integral operator of the same rearrangement of its gradient (note that these rearrangements are noincreasing functions of one variable).

The paper is organized as follows. In Section 2 we introduce basic notation, recall important auxiliary results and formulate the main results. An alternative approach to Lam's one giving the exact value of the Moser constant and the corresponding embedding theorems are presented in Section 3. In Section 4 we show that the Moser constant is sharp (proof of this part is analogical to that of Lam) and we derive optimality of the exponential embedding as well as its non compactness. Finally, in Section 5 we derive an analogue of Lions' principle of concentrated compactness for the weighted case $\mu = m_A$. The last Section 6 is devoted to some concluding remarks.

2 Basic definitions and main results

We start this section with basic definitions and necessary preliminary results.

Measure space. By the symbol (X,μ) we denote a measure space X with a nonnegative σ-measure μ. If Ω is a μ-measurable subset of X, we denote by $\mu(\Omega)$ the μ-measure of Ω, that is, $\mu(\Omega) = \int_\Omega \,\mathrm{d}\mu(x)$. If $X = \mathbb{R}^n$ with its n-dimensional Lebesgue measure $\mathrm{d}\mu(x) = \mathrm{d}x$ and $M \subset \mathbb{R}^n$ is a Lebesgue measurable set in $\mathbb{R}^n$, we use the notation $|M| = \int_\Omega \,\mathrm{d}x$.

Lebesgue space. If (X,μ) is a measure space, we denote by $L^p(\Omega,\mu)$, $p \in [1,\infty]$, the *Lebesgue space*, of all μ-measurable functions f on a μ-measurable set Ω in X, equipped with the norm

$$\|f\|_{p,\Omega,\mu} = \begin{cases} \left(\int_\Omega |f(x)|^p \,\mathrm{d}\mu(x)\right)^{1/p} & \text{if } p < \infty, \\ \mu\text{-ess sup}_{x\in\Omega} |f(x)| & \text{if } p = \infty. \end{cases}$$

If $X = \mathbb{R}^n$ and μ is the n-dimensional Lebesgue measure, then we write $\|\cdot\|_{p,\Omega}$ instead of $\|\cdot\|_{p,\Omega,\mu}$. Moreover, we simply write $\|\cdot\|_{p,\mu}$ (or $\|\cdot\|_p$) when $\Omega = X$ (or $\Omega = \mathbb{R}^n$). For $p \in [1,\infty]$ we define the *Hölder conjugate* number $p' \in [1,\infty]$ by the equality $\frac{1}{p} + \frac{1}{p'} = 1$.

Convergence in measure. Let Ω be a μ-measurable set in the measure space (X,μ). We say that the sequence of μ-measurable functions $\{u_k\}_{k=1}^\infty$ *converges in measure* μ to a function u on Ω, write $u_k \xrightarrow{\mu} u$ on Ω, if, for any $\varepsilon > 0$,

$$\lim_{k\to\infty} \mu\big(\{x \in \Omega\,;\ |u_k(x) - u(x)| \geq \varepsilon\}\big) = 0. \tag{2.1}$$

Lemma 2.1. *Let Ω be a μ-measurable set in the measure space (X,μ) such that $\mu(\Omega) < \infty$. Suppose that a sequence $\{u_k\}_{k=1}^\infty$ converges to u in $L^1(\Omega,\mu)$. Then $u_k \xrightarrow{\mu} u$ on Ω.*

Proof. Recall the Markov inequality $\mu\big(\{x \in X\,;\ f(x) \geq \varepsilon\}\big) \leq \frac{1}{\varepsilon} \int_X f(x) \,\mathrm{d}\mu(x)$ (where $\varepsilon > 0$ and $f \geq 0$). Applying this inequality to $f(x) = |u_k(x) - u(x)|$, $x \in \Omega$, with a fixed $\varepsilon > 0$ we obtain $\mu\big(\{x \in \Omega\,;\ |u_k(x) - u(x)| \geq \varepsilon\}\big) \leq \frac{1}{\varepsilon} \int_\Omega |u_k(x) - u(x)| \,\mathrm{d}\mu(x)$ and (2.1) follows. □

Orlicz space. A function Φ is called a *Young function* if it is continuous, non-negative, strictly increasing and convex on $[0,\infty)$ such that $\lim\limits_{t\to 0_+} \Phi(t)/t = \lim\limits_{t\to\infty} t/\Phi(t) = 0$ (such a function is usually called an N-function). Let Ω be a measurable set in (X,μ). The *Orlicz space* $L^\Phi(\Omega,\mu)$ with a Young function Φ is the set of all μ-measurable functions f on Ω equipped with the *Luxemburg norm*

$$\|f\|_{\Phi,\Omega,\mu} = \inf\big\{\lambda > 0\,;\ \textstyle\int_\Omega \Phi\big(\frac{|f(x)|}{\lambda}\big) \,\mathrm{d}\mu(x) \leq 1\big\}. \tag{2.2}$$

Another norm, equivalent to (2.2), is called the *Orlicz norm*

$$|||f|||_{\Phi,\Omega,\mu} = \sup_{g} \int_{\Omega} f(x)\, g(x)\, \mathrm{d}\mu(x) \tag{2.3}$$

with the supremum taken over all μ-measurable functions g such that

$$\int_{\Omega} \Psi\big(|g(x)|\big)\, \mathrm{d}\mu(x) \le 1, \tag{2.4}$$

where Ψ is the *complementary* Young function to Φ (for more details see e.g. [20, Sections 4.2 and 4.3]). We shall need the *Young inequality* (its detailed proof can be found e.g. in [27, Paragraph 4 of Chapter 5]).

Proposition 2.2. *Let Φ and Ψ be a pair of complementary Young functions. Then, for all $a, b \ge 0$,*

$$ab \le \Phi(a) + \Psi(b).$$

Together with (2.3) and (2.4) this inequality immediately implies that

$$|||f|||_{\Phi,\Omega,\mu} \le \int_{\Omega} \Phi\big(|f(x)|\big)\, \mathrm{d}\mu(x) + 1. \tag{2.5}$$

Finally, let us note that $L^{\Phi}(\Omega,\mu)$ (with any of the norms (2.2), (2.3)) is a Banach space.

Remark 2.3. Note that the Lebesgue space $L^{p}(\Omega,\mu)$, $p \in (1,\infty)$, coincides with the Orlicz space $L^{\Phi}(\Omega,\mu)$ with Young function $\Phi(t) = t^{p}$, $t \ge 0$.

Embeddings. Given two Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural *embedding* $\mathrm{id} : X \to Y$ is continuous, that is, there is a positive constant C such that

$$\|f\|_{Y} \le C\, \|f\|_{X} \quad \text{for all } f \in X$$

(here $\|f\|_{X}$ and $\|f\|_{Y}$ denote norms of f in X and f in Y, respectively). We say that the embedding is *compact* if the mapping $\mathrm{id} : X \to Y$ compact, that is, the unit ball of X is a relatively compact set in Y, we use the notation $X \hookrightarrow\hookrightarrow Y$.

Embedding properties of Banach function spaces. Lebesgue spaces $L^{p}(\Omega,\mu)$ as well as Orlicz spaces $L^{\Phi}(\Omega,\mu)$ belong to the class of so called *Banach function spaces*. Theorem 1.8 of [3, Chapter 1] claims that if X and Y are Banach function space over the same measure space, then $X \subset Y$ is equivalent to $X \hookrightarrow Y$. In particular,

$$L^{\Phi_1}(\Omega,\mu) \subset L^{\Phi_2}(\Omega,\mu) \quad \Longleftrightarrow \quad L^{\Phi_1}(\Omega,\mu) \hookrightarrow L^{\Phi_2}(\Omega,\mu). \tag{2.6}$$

Sobolev space. Let Ω be a domain in a measure space $(\mathbb{R}^n, \mu)$ with a nonnegative Borel measure μ and let $p \in [1, \infty]$. The *Sobolev space* $W^{1,p}(\Omega, \mu)$ is defined as the set

$$W^{1,p}(\Omega, \mu) = \left\{u;\ u, \tfrac{\partial}{\partial x_i} u \in L^p(\Omega, \mu) \text{ if } i = 1, \ldots, n\right\}$$

equipped with the norm

$$\|u\| = \|u\|_{p,\Omega,\mu} + \|\nabla u\|_{p,\Omega,\mu}, \tag{2.7}$$

where ∇u is the gradient of u and $|\nabla u|$ is its Euclidean length, that is,

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right), \quad |\nabla u| = \left(\sum_{j=1}^{n} (\partial u / \partial x_j)^2\right)^{1/2}$$

(to simplify the notation we are writing $\|\nabla u\|_{p,\Omega,\mu}$ instead of $\|\,|\nabla u|\,\|_{p,\Omega,\mu}$). We denote by $W_0^{1,p}(\Omega, \mu)$ the closure of $C_0^\infty(\Omega)$ (the space of infinitely differentiable functions with compact support) in $W^{1,p}(\Omega, \mu)$ with respect to the norm (2.7).

Remark 2.4. It is known that if $p \in (1, \infty)$ then the space $W_0^{1,p}(\Omega, \mu)$ is a reflexive Banach space. Moreover, if μ is the Lebesgue measure, then $\|\nabla u\|_{p;\Omega}$ and (2.7) are equivalent norms on the space $W_0^{1,p}(\Omega)$ (see Theorem 2.8 below for more general case).

Monomial weight. Let m_A be a monomial measure defined by (1.6) and (1.7). Thus, if E is a Lebesgue measurable set in $\mathbb{R}^n$, then

$$m_A(E) = \int_E x^A \,\mathrm{d}x. \tag{2.8}$$

Ball in $\mathbb{R}^n$. By the symbol $B(x, R)$ we denote the n-dimensional ball centered at $x \in \mathbb{R}^n$ with radius $R > 0$, that is

$$B(x, R) = \{y \in \mathbb{R}^n;\ |x - y| < R\}.$$

We use the notation $B = B(0, 1)$. It is easy to see that (cf. [12])

$$D\,m_A(B) = P_A(B),$$

where m_A is defined in (2.8) and

$$P_A(B) = \int_{\partial B} x^A \,\mathrm{d}\sigma(x), \tag{2.9}$$

where σ denotes the $(n-1)$-dimensional Hausdorff measure. Moreover (cf. [4, Lemma 4.1]),

$$m_A(B) = \frac{\Gamma\left(\frac{A_1+1}{2}\right)\Gamma\left(\frac{A_2+1}{2}\right)\cdots\Gamma\left(\frac{A_n+1}{2}\right)}{\Gamma\left(\frac{D}{2}+1\right)}.$$

Remark 2.5. If $A_1 = \cdots = A_n = 0$, we use the symbols s_n and ω_{n-1} for the volume of the unit ball B and for the surface area of the unit sphere $S_{n-1} = \{y \in \mathbb{R}^n;\ |y| = 1\}$, respectively. That is,

$$\omega_{n-1} = n\,s_n = n\,\pi^{n/2} / \Gamma\left(\tfrac{n}{2} + 1\right). \tag{2.10}$$

Space of Radon measures. We shall need the following result (cf. [10, Corollary 7.18]).

Proposition 2.6. *Let G be a bounded domain in $\mathbb{R}^n$. Then $C(\overline{G})'$, the topological dual to the space of all continuous functions $C(\overline{G})$ on $\overline{G}$, is isometrically isomorphic to the space of Radon measures $\mathscr{M}(\overline{G})$.*

Main results. We start with a result by Lam [12]. We present an alternative proof of the Trudinger-Moser type inequality as well as proofs of corresponding embeddings in Section 3.

Theorem 2.7. *Assume that Ω is a domain in $\mathbb{R}^n$ such that $m_A(\Omega) < \infty$. Let $u \in C_0^\infty(\Omega)$ be such that*

$$\int_\Omega |\nabla u(x)|^D x^A \, \mathrm{d}x \le 1.$$

Let

$$0 < \alpha \le \alpha_{D,A} = D P_A(B)^{1/(D-1)}. \tag{2.11}$$

Then there exists a constant $c_0 > 0$ independent of u and Ω such that

$$\frac{1}{m_A(\Omega)} \int_\Omega \exp\left(\alpha\, |u(x)|^{D'}\right) x^A \, \mathrm{d}x \le c_0, \tag{2.12}$$

Our first result is a Poncaré inequality. Our proof of this inequality is based on rearrangements and weighted Hardy-type inequalities.

Theorem 2.8. *Assume that Ω is a domain in $\mathbb{R}^n$ such that $m_A(\Omega) < \infty$. Then there is a constant $c > 0$ such that*

$$\|u\|_{D,\Omega,m_A} \le c \|\nabla u\|_{D,\Omega,m_A}, \tag{2.13}$$

for every $u \in W_0^{1,D}(\Omega, m_A)$. In particular the quantity $\|\nabla u\|_{D,\Omega,m_A}$ is an equivalent norm to (2.7) *in the Sobolev space $W_0^{1,D}(\Omega, m_A)$.*

The previous two theorems immediately imply the following embedding.

Corollary 2.9. *Let Ω be a domain in $\mathbb{R}^n$ such that $m_A(\Omega) < \infty$. Then*

$$W_0^{1,D}(\Omega, m_A) \hookrightarrow L^\Phi(\Omega, m_A), \tag{2.14}$$

where $\Phi(t) = \exp(t^{D'}) - 1$, $t \ge 0$.

The next theorem is about compact embeddings.

Theorem 2.10. *Let Ω be a domain in $\mathbb{R}^n$ with $m_A(\Omega) < \infty$ and such that embedding* (2.14) *takes place. Suppose that $L^\Psi(\Omega, m_A)$ is an Orlicz space satisfying*

$$L^\Phi(\Omega, m_A) \subsetneq L^\Psi(\Omega, m_A).$$

Then

$$W_0^{1,D}(\Omega, m_A) \hookrightarrow\hookrightarrow L^\Psi(\Omega, m_A). \tag{2.15}$$

Remark 2.11. It follows immediately from Theorem 2.10 that if Ω is a domain in $\mathbb{R}^n$ with $m_A(\Omega) < \infty$, then

$$W_0^{1,D}(\Omega, m_A) \hookrightarrow\hookrightarrow L^D(\Omega, m_A). \tag{2.16}$$

In the next theorem we claim that embedding (2.14) is optimal. For a proof of the result we employ a system of test functions used for proving sharpness of the constant $\alpha_{D,A}$. The proof is given in Section 4.

Theorem 2.12. *Let Ω be a domain in $\mathbb{R}^n$ such that with $m_A(\Omega) < \infty$, and $0 \in \Omega$ if $A_i > 0$ for some $i \in \{1,\dots,n\}$. Then*

(i) *the space $W_0^{1,D}(\Omega, m_A)$ is not embedded to any Orlicz space $L^\Psi(\Omega, m_A)$ such that $L^\Psi(\Omega, m_A) \subsetneq L^\Phi(\Omega, m_A)$,*
(ii) *embedding* (2.14) *is not compact.*

Since, under the assumptions of Theorem 2.12, embedding (2.14) is not compact, it is natural to ask why some weaker form of compactness can be true. The answer is positive, we prove (in Section 5) an analogue of the *concentrated compactness principle* of P.-L. Lions [15].

Theorem 2.13. *Let Ω be a bounded domain in $\mathbb{R}^n$ such that $0 \in \Omega$ if $A_i > 0$ for some $i \in \{1,\dots,n\}$. Let $\{u_k\}_{k=1}^\infty \subset W_0^{1,D}(\Omega, m_A)$ be such that $\int_\Omega |\nabla u_k(x)|^D x^A \,\mathrm{d}x \le 1$. Moreover, suppose that*

$$u_k \rightharpoonup u \text{ in } W_0^{1,D}(\Omega, m_A), \quad u_k \to u \text{ a.e. in } \Omega, \quad \text{and} \quad |\nabla u_k(x)|^D x^A \overset{*}{\rightharpoonup} \nu \text{ in } \mathscr{M}(\overline{\Omega}). \tag{2.17}$$

Then one of the following possibilities takes place.

(i) *If $u = 0$, $\nu = \delta_{x_0}$ for some $x_0 \in \overline{\Omega}$, and*

$$\int_\Omega \exp\left(\alpha_{D,A} |u_k(x)|^{D'}\right) x^A \,\mathrm{d}x \to c + m_A(\Omega)$$

for some $c \ge 0$, then

$$\exp\left(\alpha_{D,A} |u_k(x)|^{D'}\right) x^A \overset{*}{\rightharpoonup} c\delta_{x_0} + m_A \restriction_\Omega \quad \text{in } \mathscr{M}\big(\overline{\Omega}\big).$$

(ii) *If $u = 0$ and ν is not a Dirac mass concentrated at one point, then there exist constants $C > 0$ and $p > 1$ such that, for all $k \in \mathbb{N}$,*

$$\int_\Omega \exp\left(p\, \alpha_{D,A} |u_k(x)|^{D'}\right) x^A \,\mathrm{d}x \le C. \tag{2.18}$$

(iii) *If $u \neq 0$, then there exist constants $C > 0$ and $p > 1$ such that* (2.18) *holds for all $k \in \mathbb{N}$.*

In each of cases (ii) *and* (iii)

$$\lim_{k\to\infty} \int_\Omega \exp\left(\alpha_{D,A} |u_k(x)|^{D'}\right) x^A \,\mathrm{d}x = \int_\Omega \exp\left(\alpha_{D,A} |u(x)|^{D'}\right) x^A \,\mathrm{d}x. \tag{2.19}$$

3 Moser constant, continuous and compact embeddings

Before we prove the results we have to do some preliminary work.

Upper estimate for the gradient We shall need the following important estimate relating u with its gradient.

Lemma 3.1. *Let* $u \in C_0^\infty(\mathbb{R}^n)$. *Then*

$$P_A(B)\,|u(x)| \le \int_{\mathbb{R}^n} |\nabla u(x-y)|\,|y|^{1-D} y^A \,\mathrm{d}y, \quad x \in \mathbb{R}^n. \tag{3.1}$$

Proof. To obtain the result we slightly modify the proof of formula (18) from [23, Chapter V]. Since $f \in C_0^\infty(\mathbb{R}^n)$ we have

$$u(x) = \int_0^\infty \nabla u(x - t\xi)\cdot \xi \,\mathrm{d}t,$$

where $\xi = (\xi_1, \dots, \xi_n)$ is an arbitrary unit vector in $\mathbb{R}^n$. Integrating this equality over the unit sphere ∂B we obtain

$$\big(\textstyle\int_{\partial B} \xi^A \,\mathrm{d}\sigma(\xi)\big) u(x) = \displaystyle\int_{\partial B}\Big(\int_0^\infty \nabla u(x-t\xi)\cdot\xi\,\mathrm{d}t\Big)\xi^A \,\mathrm{d}\sigma(\xi).$$

Thus, applying the Fubini theorem and the change of variables $y = t\xi$,

$$\begin{aligned}
\big(\textstyle\int_{\partial B} \xi^A \,\mathrm{d}\sigma(\xi)\big)\,|u(x)| &\le \displaystyle\int_{\partial B}\Big(\int_0^\infty |\nabla u(x-t\xi)|\,\mathrm{d}t\Big)\xi^A \,\mathrm{d}\sigma(\xi) \\
&= \int_0^\infty \Big(\int_{\partial B} |\nabla u(x-t\xi)|\xi^A \,\mathrm{d}\sigma(\xi)\Big)\,\mathrm{d}t = \int_0^\infty \Big(\int_{\partial B(0,t)} |\nabla u(x-y)| y^A \,\mathrm{d}\sigma(y)\Big) t^{1-D}\,\mathrm{d}t \\
&= \int_{\mathbb{R}^n} |\nabla u(x-y)|\,|y|^{1-D} y^A \,\mathrm{d}y.
\end{aligned}$$

□

Rearrangement Let f be a measurable function on the measure space (X, μ). We define its *nonincreasing rearrangement* f_μ^* (on the interval $(0,\infty)$) by

$$f_\mu^*(t) = \inf\{s > 0;\ \lambda(s) \le t\}, \quad \text{where} \quad \lambda(s) = \mu\big(\{x \in X;\ |f(x)| > s\}\big).$$

We also define the corresponding *maximal operator*

$$f_\mu^{**}(t) = \frac{1}{t}\int_0^t f_\mu^*(s)\,\mathrm{d}s, \quad t > 0.$$

Remark 3.2. It is easy to see that, if Ψ is a continuous and increasing function, then

$$\int_\Omega \Psi\big(|u(x)|\big)\,\mathrm{d}\mu(x) = \int_0^{\mu(\Omega)} \Psi\big(u^*(t)\big)\,\mathrm{d}t.$$

It implies that $\|u\|_{\Phi,\Omega,\mu} = \|u^*\|_{\Phi,(0,\mu(\Omega))}$ and, for $p \in [1,\infty)$, $\|u\|_{p,\Omega,\mu} = \|u^*\|_{p,(0,\mu(\Omega))}$.

Rearrangements of I_1 in the measure space $(\mathbb{R}^n, m_A)$ Set

$$I_1(x) = |x|^{1-D}, \quad x \in \mathbb{R}^n. \tag{3.2}$$

Let us derive $(I_1)_{m_A}^*$ and $(I_1)_{m_A}^{**}$ with respect to the measure m_A from (2.8). Observe that

$$\lambda(s) = m_A\big(\{x \in \mathbb{R}^n;\ |x|^{1-D} > s\}\big) = \int_{B(0,s^{1/(1-D)})} x^A\,\mathrm{d}x = s^{-D'}\int_B y^A\,\mathrm{d}y = s^{-D'} m_A(B).$$

Thus,

$$(I_1)_{m_A}^*(t) = \Big(\frac{m_A(B)}{t}\Big)^{1/D'}, \quad (I_1)_{m_A}^{**}(t) = D\Big(\frac{m_A(B)}{t}\Big)^{1/D'}, \quad t > 0. \tag{3.3}$$

O'Neil inequality We make use of the following result of O'Neil [18].

Theorem 3.3. *Let f and g be two μ-measurable functions on a measure space (X,μ). Then*

$$(f*g)^{**}_{\mu}(t) \le t\, f^{**}_{\mu}(t)\, g^{**}_{\mu}(t) + \int_t^{\infty} f^*_{\mu}(s)\, g^*_{\mu}(s)\,\mathrm{d}s, \quad t>0, \tag{3.4}$$

where the convolution $f*g$ is considered with respect to the measure μ, that is

$$(f*g)(x) = \int_X f(x-y)g(y)\,\mathrm{d}\mu(y).$$

Proof. Set

$$T(f,g)(x) = (f*g)(x) = \int_X f(x-y)g(y)\,\mathrm{d}\mu(y). \tag{3.5}$$

Observe that T is a convolution operator in the sense of [18, Definition 1.1]. That means that it satisfies the following properties:

$$T(f_1+f_2,g) = T(f_1,g)+T(f_2,g), \quad T(f,g_1+g_2) = T(f,g_1)+T(f,g_2),$$
$$\|T(f,g)\|_{\infty,\mu} \le \|f\|_{1,\mu}\,\|g\|_{\infty,\mu}, \|T(f,g)\|_{\infty,\mu} \le \|f\|_{\infty,\mu}\,\|g\|_{1,\mu}, \|T(f,g)\|_{1,\mu} \le \|f\|_{1,\mu}\,\|g\|_{1,\mu}.$$

The first two identities are obvious. As concerns the third property,

$$\|T(f,g)\|_{\infty,\mu} \le \int_X |f(x-y)g(y)|\,\mathrm{d}\mu(y) \le \|g\|_{\infty,\mu}\int_X |f(x-y)|\,\mathrm{d}\mu(y) = \|f\|_{1,\mu}\,\|g\|_{\infty,\mu},$$

the fourth one follows by the same argument. It remains to verify the last property. We have

$$\begin{aligned}\|T(f,g)\|_{1,\mu} &\le \int_X \mathrm{d}\mu(x)\int_X |f(x-y)g(y)|\,\mathrm{d}\mu(y) = \int_X |g(y)|\Big(\int_X |f(x-y)|\,\mathrm{d}\mu(x)\Big)\,\mathrm{d}\mu(y)\\ &= \int_X |g(y)|\Big(\int_X |f(x)|\,\mathrm{d}\mu(x)\Big)\,\mathrm{d}\mu(y) = \|f\|_{1,\mu}\,\|g\|_{1,\mu}.\end{aligned}$$

Inequality (3.4) follows from [18, Theorem 1.7]. □

Proof of Theorem 2.7. For the sake of simplicity we put

$$\mu = m_A, \quad f = |\nabla u|$$

and write u^*, f^* and f^{**} instead of u^*_{μ}, f^*_{μ} and f^{**}_{μ}, respectively.

By (3.1), (3.4) and (3.3) we obtain, for $\alpha \in (0,\alpha_{D,A}]$,

$$\begin{aligned}\alpha^{1/D'}u^*(\tau) &\le \big(D P_A(B)^{1/(D-1)}\big)^{(D-1)/D}u^*(\tau) \le D^{1/D'}P_A(B)^{-1/D'}(I_1*f)^*(\tau)\\ &\le D\,\tau^{-\frac{D-1}{D}}\int_0^{\tau} f^*(s)\,\mathrm{d}s + \int_{\tau}^{\mu(\Omega)} f^*(s)s^{-\frac{D-1}{D}}\,\mathrm{d}s, \quad \tau>0.\end{aligned} \tag{3.6}$$

Using the change of variables $\tau = \mu(\Omega)e^{-t}$ we obtain

$$\begin{aligned}\frac{1}{\mu(\Omega)}\int_{\Omega}\exp\big(\alpha\,|u(x)|^{D'}\big)x^A\,\mathrm{d}x &= \frac{1}{\mu(\Omega)}\int_0^{\mu(\Omega)}\exp\big(\alpha\,u^*(\tau)^{D'}\big)\,\mathrm{d}\tau\\ &= \int_0^{\infty}\exp\big(\alpha\,u^*\big(\mu(\Omega)e^{-t}\big)^{D'}\big)\,e^{-t}\,\mathrm{d}t.\end{aligned}$$

Consequently, by (3.6), where we made the change of variables $s=\mu(\Omega)e^{-y}$ at the integrals on the right hand side

$$\begin{aligned}
&\frac{1}{\mu(\Omega)}\int_\Omega \exp\left(\alpha\,|u(x)|^{D'}\right)x^A\,\mathrm{d}x\\
&=\int_0^\infty \exp\left(\left(D\left(\mu(\Omega)e^{-t}\right)^{-\frac{D-1}{D}}\int_0^{\mu(\Omega)e^{-t}} f^*(s)\,\mathrm{d}s+\int_{\mu(\Omega)e^{-t}}^{\mu(\Omega)} f^*(s)s^{-\frac{D-1}{D}}\,\mathrm{d}s\right)^{D'}\right)e^{-t}\,\mathrm{d}t\\
&\qquad\le\int_0^\infty \exp\left(\left(D\mu(\Omega)^{\frac{1}{D}}\,e^{t\frac{D-1}{D}}\int_t^\infty f^*(\mu(\Omega)\,e^{-y})\,e^{-y}\,\mathrm{d}y\right.\right.\\
&\qquad\qquad\left.\left.+\mu(\Omega)^{\frac{1}{D}}\int_0^t f^*(\mu(\Omega)\,e^{-y})\,e^{y\frac{D-1}{D}}\,e^{-y}\,\mathrm{d}y\right)^{D'}-t\right)\mathrm{d}t
\end{aligned}$$

Thus,

$$\frac{1}{\mu(\Omega)}\int_\Omega \exp\left(\alpha\,|u(x)|^{D'}\right)x^A\,\mathrm{d}x=\int_0^\infty \exp(-F(t))\,\mathrm{d}t, \tag{3.7}$$

where

$$\begin{aligned}
F(t):=t-\Big(D\mu(\Omega)^{\frac{1}{D}}\,e^{t\frac{D-1}{D}}&\int_t^\infty f^*(\mu(\Omega)\,e^{-y})\,e^{-y}\,\mathrm{d}y\\
&+\mu(\Omega)^{\frac{1}{D}}\int_0^t f^*(\mu(\Omega)\,e^{-y})\,e^{y\frac{D-1}{D}}\,e^{-y}\,\mathrm{d}y\Big)^{D'}.
\end{aligned}\tag{3.8}$$

We write the integral on the right-hand side of (3.7) as

$$\int_0^\infty \exp(-F(t))\,\mathrm{d}t=\int_{-\infty}^\infty \left|E_\lambda\right|e^{-\lambda}\,\mathrm{d}\lambda, \tag{3.9}$$

where

$$E_\lambda:=\{t\ge 0\,;\,F(t)\le\lambda\}, \tag{3.10}$$

and we prove that

$$\text{there is } \lambda_0 \text{ such that}\quad \lambda<\lambda_0 \quad\Longrightarrow\quad E_\lambda=\emptyset, \tag{3.11}$$

$$\text{there are } A_1,A_2>0 \text{ independent of } u \text{ such that } |E_\lambda|\le A_1|\lambda|+A_2 \text{ for all } \lambda\ge\lambda_0. \tag{3.12}$$

To verify (3.11) we first observe that

$$F \text{ is a continuous increasing function on } [0,\infty). \tag{3.13}$$

The continuity is clear. As concerns its monotonicity, we have

$$F(t)=t-(\mu(\Omega)^{\frac{1}{D}}\,F_0(t))^{D'},$$

where

$$F_0(t)=D\,e^{t\frac{n-1}{D}}\int_t^\infty f^*(\mu(\Omega)\,e^{-y})\,e^{-y}\,\mathrm{d}y+\int_0^t f^*(\mu(\Omega)\,e^{-y})\,e^{-y\frac{1}{D}}\,\mathrm{d}y,$$

and

$$\begin{aligned}
F_0'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(D\,e^{t\,\frac{D-1}{D}}\int_t^\infty f^*(\mu(\Omega)\,e^{-y})\,e^{-y}\,\mathrm{d}y + \int_0^t f^*(\mu(\Omega)\,e^{-y})\,e^{-y\frac{1}{D}}\,\mathrm{d}y\Big)\\
&= -D\,e^{t\,\frac{n-1}{D}} f^*(\mu(\Omega)\,e^{-t})\,e^{-t} + (D-1)\,e^{t\,\frac{D-1}{D}}\int_t^\infty f^*(\mu(\Omega)\,e^{-y})\,e^{-y}\,\mathrm{d}y\\
&\qquad + e^{t\,\frac{D-1}{D}} f^*(\mu(\Omega)\,e^{-t})\,e^{-t}\\
&= (D-1)\,e^{-t\,\frac{1}{D}}\Big(e^t\int_t^\infty f^*(\mu(\Omega)\,e^{-y})\,e^{-y}\,\mathrm{d}y - f^*(\mu(\Omega)\,e^{-t})\Big) \le 0,
\end{aligned}$$

since

$$e^t\int_t^\infty f^*(\mu(\Omega)\,e^{-y})\,e^{-y}\,\mathrm{d}y \le e^t f^*(\mu(\Omega)\,e^{-t})\int_t^\infty e^{-y}\,\mathrm{d}y = f^*(\mu(\Omega)\,e^{-t}).$$

Consequently, the first derivative of F,

$$F'(t) = 1 - D'\mu(\Omega)^{\frac{1}{D}}\,(\mu(\Omega)^{\frac{1}{D}}\,F_0(t))^{D'-1}F_0'(t),$$

is positive on $(0,\infty)$ and so, the function F is (strictly) increasing on $(0,\infty)$.

For $t \ge 0$, we obtain, by the Hölder inequality with $\frac{1}{D}+\frac{1}{D'}=1$,

$$\begin{aligned}
&D\,\mu(\Omega)^{\frac{1}{D}}\,e^{t\,\frac{D-1}{D}}\int_t^\infty f^*(\mu(\Omega)\,e^{-y})\,e^{-y}\,\mathrm{d}y\\
&\qquad \le D e^{t\,\frac{D-1}{D}}\Big(\int_t^\infty e^{-y}\,\mathrm{d}y\Big)^{1/D'}\Big(\int_t^\infty \big(f^*(\mu(\Omega)\,e^{-y})\big)^D\mu(\Omega)\,e^{-y}\,\mathrm{d}y\Big)^{1/D}. \qquad (3.14)
\end{aligned}$$

Observe that

$$1 \ge \int_\Omega |\nabla u(x)|^D x^A\,\mathrm{d}x = \int_0^\infty \big(f^*(\mu(\Omega)\,e^{-y})\big)^D\mu(\Omega)\,e^{-y}\,\mathrm{d}y. \qquad (3.15)$$

By (3.8), (3.13), (3.14), the obvious fact that $\int_0^\infty e^{-y}\,\mathrm{d}y = 1$ and (3.15), we have, for every $t \ge 0$,

$$F(t) \ge F(0) = -\Big(D\,\mu(\Omega)^{\frac{1}{D}}\int_0^\infty f^*(\mu(\Omega)\,e^{-y})\,e^{-y}\,\mathrm{d}y\Big)^{D'} \ge -D^{D'} = \lambda_0,$$

which implies (3.11).

We prove (3.12). Let $0 \le t_1 < t_2 < \infty$. By the Hölder inequality with $\frac{1}{D}+\frac{1}{D'}=1$ we obtain

$$\begin{aligned}
&\int_{t_1}^{t_2} f^*(\mu(\Omega)\,e^{-y})\,\big(\mu(\Omega)\,e^{-y}\big)^{\frac{1}{D}}\,\mathrm{d}y\\
&\qquad \le \Big(\int_{t_1}^{t_2}\big(f^*(\mu(\Omega)\,e^{-y})\big)^D\mu(\Omega)\,e^{-y}\,\mathrm{d}y\Big)^{1/D}\big(t_2-t_1\big)^{1/D'}. \qquad (3.16)
\end{aligned}$$

Since $\int_t^\infty e^{-y}\,\mathrm{d}y = e^{-t}$, $t > 0$, estimate (3.14) imply that, for every $t > 0$,

$$\begin{aligned}
&D\,\mu(\Omega)^{\frac{1}{D}}\,e^{t\,\frac{D-1}{D}}\int_t^\infty f^*(\mu(\Omega)\,e^{-y})\,e^{-y}\,\mathrm{d}y\\
&\qquad \le D e^{t\,\frac{D-1}{D}}\Big(\int_t^\infty e^{-y}\,\mathrm{d}y\Big)^{1/D'}\Big(\int_t^\infty \big(f^*(\mu(\Omega)\,e^{-y})\big)^D\mu(\Omega)\,e^{-y}\,\mathrm{d}y\Big)^{1/D}\\
&\qquad \le D\Big(\int_t^\infty \big(f^*(\mu(\Omega)\,e^{-y})\big)^D\mu(\Omega)\,e^{-y}\,\mathrm{d}y\Big)^{1/D}. \qquad (3.17)
\end{aligned}$$

For $t > 0$ set

$$g(t) = \Big(\int_t^\infty \big(f^*(\mu(\Omega)\, e^{-y}) \big)^D \mu(\Omega)\, e^{-y}\, \mathrm{d}y \Big)^{1/D}. \tag{3.18}$$

Then (3.15) implies that

$$0 \le g(t) \le 1 \quad \text{and} \quad 0 \le \Big(\int_0^t \big(f^*(\mu(\Omega)\, e^{-y}) \big)^D \mu(\Omega)\, e^{-y}\, \mathrm{d}y \Big)^{1/D} \le 1, \quad t > 0. \tag{3.19}$$

Let $\lambda > 0$, $t_1, t_2 \in E_\lambda$, $t_1 < t_2$. Then, using (3.8), (3.17), (3.16), (3.18) and (3.19), we obtain

$$\begin{aligned} t_2 - \lambda &\le \Big(D\, \mu(\Omega)^{\frac{1}{D}}\, e^{t\frac{D-1}{D}} \int_{t_2}^\infty f^*(\mu(\Omega)\, e^{-y})\, e^{-y}\, \mathrm{d}y + \mu(\Omega)^{\frac{1}{D}} \int_0^{t_2} f^*(\mu(\Omega)\, e^{-y})\, e^{-y}\, \mathrm{d}y \Big)^{D'} \\ &\le \Big(D\, g(t_2) + \mu(\Omega)^{\frac{1}{D}} \Big(\int_0^{t_1} \cdots\, \mathrm{d}y + \int_{t_1}^{t_2} \cdots\, \mathrm{d}y \Big) \Big)^{D'} \\ &\le \Big(D\, g(t_2) + t_1^{1/D'} + (t_2 - t_1)^{1/D'}\, g(t_1) \Big)^{D'}. \end{aligned} \tag{3.20}$$

Thus,

$$t_2 - \lambda \le \Big(D\, g(t_2) + t_1^{1/D'} + (t_2 - t_1)^{1/D'}\, g(t_1) \Big)^{D'}. \tag{3.21}$$

Assume that $\lambda \ge D^{D'}$, $t_1 = \lambda$ (obviously $\lambda \in E_\lambda$) and $t_2 = (\theta + 1)\lambda$ with $\theta > 0$. Then, by (3.21) and (3.19), $\lambda\, \theta \le \lambda \Big(2 + \theta^{1/D'}\, g(\lambda) \Big)^{D'}$. Thus, $\theta^{1/D'} \le 2 + \theta^{1/D'}\, g(\lambda)$. Since $\lim_{t\to\infty} g(t) = 0$, we deduce from the last inequality, that if $\lambda \ge \lambda_1$, where $\lambda_1 \ge D^{D'}$ is such that $g(\lambda_1) < 1$, then θ is bounded, and consequently (3.12) follows for all $\lambda \ge \lambda_1$. If $\lambda \in [\lambda_0, \lambda_1)$ we use the inclusion $E_\lambda \subset E_{\lambda_1}$, which implies $|E_\lambda| \le |E_{\lambda_1}| \le A_1 + A_2\lambda_1$, and (3.12) is verified (with the constant $(A_1 + A_2\lambda_1)$ instead of A_1) for all $\lambda \ge \lambda_0$.

To finish the proof of (2.12) we use (3.7), (3.9), (3.11) and (3.12) to get

$$\frac{1}{\mu(\Omega)} \int_\Omega \exp\big(\alpha\, |u(x)|^{D'} \big) x^A\, \mathrm{d}x = \int_{\lambda_0}^\infty \big| E_\lambda \big|\, e^{-\lambda}\, \mathrm{d}\lambda \le \int_{\lambda_0}^\infty (A_1 + A_2|\lambda|)\, e^{-\lambda}\, \mathrm{d}\lambda = c_0 < \infty.$$

□

Now, we outline our proof of Poincaré inequality (2.13).

Proof of Theorem 2.8. It is enough to prove inequality (2.13) for $u \in C_0^\infty(\Omega)$. Using rearrangements, we easily see that our task is to prove that there exists a constant $c > 0$ such that, for all $u \in C_0^\infty(\Omega)$,

$$\int_0^{m_A(\Omega)} u^*(t)^D\, \mathrm{d}t \le c \int_0^{m_A(\Omega)} f^*(t)^D\, \mathrm{d}t, \tag{3.22}$$

where u^* and f^* have the same meaning as in the proof of Theorem 2.7. We rewrite (3.6) to the form

$$u^*(t) \le c_1 \Big(t^{-\frac{D-1}{D}} \int_0^t f^*(s)\, \mathrm{d}s + \int_t^{m_A(\Omega)} f^*(s) s^{-\frac{D-1}{D}}\, \mathrm{d}s \Big), \quad t > 0,$$

where $c_1 > 0$ is a constant independent of u. To prove (3.22) it is sufficient to show that the following Hardy-type inequalities

$$\int_0^{m_A(\Omega)} \Big(t^{-\frac{D-1}{D}} \int_0^t f^*(s)\,\mathrm{d}s\Big)^D \mathrm{d}t \le c_2 \int_0^{m_A(\Omega)} f^*(t)^D\,\mathrm{d}t,$$
$$\int_0^{m_A(\Omega)} \Big(\int_t^{m_A(\Omega)} f^*(s) s^{-\frac{D-1}{D}}\,\mathrm{d}s\Big)^D \le c_3 \int_0^{m_A(\Omega)} f^*(t)^D\,\mathrm{d}t$$

(with constants $c_2 > 0$, $c_3 > 0$ independent of f) hold. These inequalities follow from the Muckenhoupt conditions (see [17]):

$$\sup_{0<R<m_A(\Omega)} \Big(\int_R^{m_A(\Omega)} t^{1-D}\,\mathrm{d}t\Big)^{1/D} \Big(\int_0^R \mathrm{d}t\Big)^{1/D'} < \infty$$

and

$$\sup_{0<R<m_A(\Omega)} \Big(\int_0^R \mathrm{d}t\Big)^{1/D} \Big(\int_R^{m_A(\Omega)} t^{-1}\,\mathrm{d}t\Big)^{1/D'} < \infty,$$

respectively. Verifying these conditions is an easy exercise, it is left to the reader. □

Proof of Corollary 2.9. From (2.12) we obtain that, for all $u \in W_0^{1,D}(\Omega, m_A)$ such that $\|\nabla u\|_{D,\Omega,m_A} \le 1$, $\int_\Omega \big(\exp(\alpha_{D,A}\,|u(x)|^{D'}) - 1\big)\,x^A\,\mathrm{d}x \le (c_0-1)/m_A(\Omega)$, that is,

$$\int_\Omega \Phi\big(\alpha\,|u(x)|\big)\,x^A\,\mathrm{d}x \le c_1,$$

where $\alpha = \alpha_{D,A}^{1/D'}$ and $c_1 = (c_0-1)/m_A(\Omega)$. Consider $L^\Phi(\Omega, m_A)$ with the Orlicz norm $|||\cdot|||_{\Phi,\Omega,m_A}$ (see (2.3)). Then, using (2.5), we obtain

$$|||\alpha\,u|||_{\Phi,\Omega,m_A} \le \int_\Omega \Phi\big(\alpha\,|u(x)|\big)\,x^A\,\mathrm{d}x + 1 \le c_1 + 1,$$

that is, $|||u|||_{\Phi,\Omega,m_A} \le (c_1+1)/\alpha$, and the embedding (2.14) follows. □

To express embeddings between Orlicz spaces we have to recall other properties of these spaces.

Definition 3.4. Let Φ_1, Φ_2 be Young functions. If there exist two positive constants c and T such that

$$\Phi_1(t) \le \Phi_2(ct) \quad \text{for all } t \ge T,$$

we say that Φ_2 *dominates* Φ_1 *near* ∞ and write

$$\Phi_1 \preceq \Phi_2.$$

If $\Phi_1 \preceq \Phi_2$ and $\Phi_2 \preceq \Phi_1$ we say that Φ_1 and Φ_2 are *equivalent* and write $\Phi_1 \approx \Phi_2$.

Remark 3.5. It is easy to see that every Young function Φ is equivalent to the function

$$\Phi_k(t) = \Phi(kt), \tag{3.23}$$

where k is an arbitrary positive number.

Theorem 3.6. *Let Φ_1 and Φ_2 be Young functions and let $\mu(\Omega) < \infty$. Then*

$$L^{\Phi_1}(\Omega,\mu) \hookrightarrow L^{\Phi_2}(\Omega,\mu) \tag{3.24}$$

if and only if $\Phi_2 \preceq \Phi_1$. In particular, if Φ is a Young function and Φ_k, $k > 0$, is defined by (3.23)*, then*

$$L^{\Phi_k}(\Omega,\mu) = L^{\Phi}(\Omega,\mu).$$

Proof. The proof can be done by the same way as for the case, where μ is the Lebesgue measure on $\mathbb{R}^n$ (see [1, Theorem 8.12]), so we omit it here. □

Definition 3.7. Let Φ_1, Φ_2 be Young functions. We say that Φ_2 *increases strictly more rapidly* than Φ_1 near ∞, write $\Phi_1 \prec\prec \Phi_2$, if

$$\lim_{t\to\infty} \frac{\Phi_2(ct)}{\Phi_1(t)} = \infty \quad \text{for every } c > 0.$$

Remark 3.8. It is not difficult to show that if Φ_1 and Φ_2 are Young functions and $\mu(\Omega) < \infty$, then

$$L^{\Phi_1}(\Omega,\mu) \supsetneqq L^{\Phi_2}(\Omega,\mu)$$

if and only if $\Phi_2 \prec\prec \Phi_1$ (cf. [20, Remark 4.5.12]).

Before proving Theorem 2.10 we need some auxiliary results.

Lemma 3.9. *Let Ω be a domain in $\mathbb{R}^n$ such that $m_A(\Omega) < \infty$. Then*

$$W_0^{1,D}(\Omega,m_A) \hookrightarrow\hookrightarrow L^1(\Omega,m_A).$$

Proof. It is sufficient to show that the set

$$K = \{u \in W_0^{1,D}(\Omega,m_A)\,;\ \|\nabla u\|_{D,\Omega,m_A} \le 1\}$$

is precompact in $L^1(\Omega,m_A)$, that is, for each $\varepsilon > 0$, the set K contains a finite ε-net. Let us fix $\varepsilon > 0$. For $\eta > 0$ set

$$\Omega^\eta = \Omega \cap \{x \in \mathbb{R}^n\,;\ |x| < 1/\eta, |x_1| > \eta, \dots, |x_n| > \eta\}, \quad \Omega_\eta = \Omega \setminus \Omega^\eta.$$

By the Hölder inequality and (**??**) we have

$$\|u\chi_{\Omega_\eta}\|_{1,\Omega,m_A} = \int_{\Omega_\eta} |u(x)|x^A\,\mathrm{d}x \le \|u\|_{D,\Omega,m_A}\Big(\int_{\Omega_\eta} x^A\,\mathrm{d}x\Big)^{D'} \le c\Big(\int_{\Omega_\eta} x^A\,\mathrm{d}x\Big)^{D'}.$$

Moreover, observe that $\Omega_\eta \subset \big(\bigcup_{i=1}^n \{x \in \Omega\,;\ |x_i| \le \eta\}\big) \setminus B(0,1/\eta)$, and so, using the assumption that $m_A(\Omega) < \infty$, $\int_{\Omega_\eta} x^A\,\mathrm{d}x \to 0$ when $\eta \to 0_+$. Thus, there exists $\eta_0 > 0$ such that, for all $\eta \in (0,\eta_0]$ and all $u \in K$,

$$\|u\chi_{\Omega_\eta}\|_{1,\Omega,m_A} < \varepsilon/3.$$

Fix $\eta \in (0,\eta_0]$ and observe that, by $W_0^{1,D}(\Omega,m_A) \hookrightarrow L^1(\Omega,m_A)$ (cf. Corollary 2.9) and by $\|u\|_{1,\Omega^\eta} \le \eta^{-(A_1+\dots+A_n)}\|u\chi_{\Omega_\eta}\|_{1,\Omega,m_A}$, the set K is a bounded set in the Sobolev space $W^{1,D}(\Omega^\eta)$. Since the embedding $W^{1,D}(\Omega^\eta)$ into $L^1(\Omega^\eta)$ is compact (see e.g. [6, Theorem 5.4.16]) we can find a finite subset $K_0 = \{u_1,\dots,u_N\} \subset K$ such that

$$\|u_k - u_j\|_{1,\Omega^\eta} < \varepsilon/(3\max_{x\in\Omega} x^A) \quad \text{for all } k,j \in \{1,\dots,N\}.$$

Finally, we obtain, for all $u_k, u_j \in K_0$, $k, j \in \{1, \dots, N\}$,

$$\begin{aligned}\|u_k - u_j\|_{1,\Omega,m_A} &\le \|u_k \chi_{\Omega_\eta}\|_{1,\Omega,m_A} + \|u_j \chi_{\Omega_\eta}\|_{1,\Omega,m_A} + \|(u_k - u_j)\chi_{\Omega^\eta}\|_{1,\Omega,m_A} \\ &\le 2\varepsilon/3 + \max_{x\in\Omega} x^A \, \|u_k - u_j\|_{1,\Omega^\eta} < \varepsilon\end{aligned}$$

and the proof is completed. □

Lemma 3.10. *Let Ω be a domain in $\mathbb{R}^n$ such that $m_A(\Omega) < \infty$. Suppose that Φ and Ψ are Young functions such that $\Psi \prec\prec \Phi$. If the sequence $\{u_k\}_{k=1}^\infty$ is bounded in $L^\Phi(\Omega, m_A)$ and $u_k \xrightarrow{m_A} u$ on Ω, then $u_k \to u$ in $L^\Psi(\Omega, m_A)$.*

Proof. The proof is analogous to [1, Proof of Theorem 8.24], and so we omit it here. □

Proof of Theorem 2.10. By Corollary 2.9 and Remark 3.8 we have

$$W_0^{1,D}(\Omega, m_A) \hookrightarrow L^\Phi(\Omega, m_A) \hookrightarrow L^\Psi(\Omega, m_A).$$

Let $\mathscr{K}$ be a bounded set in $W_0^{1,D}(\Omega, m_A)$. By Lemma 3.9 there is a sequence $\{u_k\}_{k=1}^\infty \subset \mathscr{K}$ and a function $u \in L^1(\Omega, m_A)$ such that $u_k \to u$ in $L^1(\Omega, m_A)$. Lemma 2.1 implies that $u_k \xrightarrow{m_A} u$ on Ω. Applying Lemma 3.10 we obtain $u_k \to u$ in $L^\Psi(\Omega, m_A)$. □

4 Optimality of the Moser constant and of the embedding

The next result can be found (including a proof) in [12, Section 3].

Lemma 4.1. *Let $n \in \mathbb{N}$, $n \ge 2$. Then, for any $\alpha > \alpha_{D,A}$,*

$$\sup_u \frac{1}{m_A(B)} \int_B \exp\left(\alpha\, |u(x)|^{D'}\right) x^A \,\mathrm{d}x = \infty. \tag{4.1}$$

where the supremum is taken over all functions $u \in C_0^\infty(B)$ satisfying

$$\int_B |\nabla u(x)|^D x^A \,\mathrm{d}x \le 1.$$

Proof. Observe that if a function f is radial, that is $f(x) = F(|x|)$, then

$$|\nabla f(x)| = \left|F'(|x|)\right| \quad \text{and} \quad \int_{B(0,R)} f(x) x^A \,\mathrm{d}x = \Big(\int_{\partial B} x^A \,\mathrm{d}\sigma(x)\Big) \int_0^R F(t)\, t^{D-1}\,\mathrm{d}t. \tag{4.2}$$

For each $r \in (0,1)$ consider the function

$$u_r(x) = \begin{cases} \Big(\int_{\partial B} x^A \,\mathrm{d}\sigma(x)\Big)^{-1/D} \big(\log(1/r)\big)^{1/D'} & \text{for } 0 \le |x| < r \\ \Big(\big(\int_{\partial B} x^A \,\mathrm{d}\sigma(x)\big)\log(1/r)\Big)^{-1/D} \log(1/|x|) & \text{for } r \le |x| \le 1 \end{cases}, \quad x \in B. \tag{4.3}$$

Then

$$|\nabla u_r(x)| = \begin{cases} 0 & \text{for } 0 \le |x| < r \\ \Big(\big(\int_{\partial B} x^A \,\mathrm{d}\sigma(x)\big)\log(1/r)\Big)^{-1/D} (1/|x|) & \text{for } r \le |x| \le 1 \end{cases}, \quad x \in B.$$

Thus,

$$\int_B |\nabla u_r(x)|^D x^A \,\mathrm{d}x = \Big(\big(\textstyle\int_{\partial B} x^A \,\mathrm{d}\sigma(x)\big)\log(1/r)\Big)^{-1}\big(\textstyle\int_{\partial B} x^A \,\mathrm{d}\sigma(x)\big)\displaystyle\int_r^1 t^{-1}\,\mathrm{d}t = 1.$$

If condition (4.1) does not hold, there is a constant $c_0 > 0$ such that

$$c_0 \ge \frac{1}{m_A(B)}\int_B \exp\big(\alpha\,|u_r(x)|^{D'}\big)x^A \,\mathrm{d}x \ge r^D \exp\Big(\alpha\,\big(\textstyle\int_{\partial B} x^A \,\mathrm{d}\sigma(x)\big)^{-1/(D-1)}\log(1/r)\Big). \tag{4.4}$$

Consequently,

$$\alpha \le \frac{D\log(1/r)+\log c_0}{\big(\int_{\partial B} x^A \,\mathrm{d}\sigma(x)\big)^{-1/(D-1)}\log(1/r)},$$

and, letting $r \to 0^+$, we obtain that $\alpha \le D\big(\int_{\partial B} x^A \,\mathrm{d}\sigma(x)\big)^{1/(D-1)} = D(P_A(B))^{1/(D-1)} = \alpha_{D,A}$ (cf. (2.11)). $\square$

Remark 4.2. Alternatively, put $r_k = \exp\big(-(\int_{\partial B} x^A \,\mathrm{d}\sigma(x))^{D'/D} k^{D'}\big)$, $k \in \mathbb{N}$. Then we can rewrite (4.3) to the form

$$u_{r_k}(x) = \begin{cases} k & \text{if } 0 \le |x| \le r_k, \\ \big(\int_{\partial B} x^A \,\mathrm{d}\sigma(x)\,k\big)^{-D'/D}\log(1/|x|) & \text{if } r_k < |x| \le 1. \end{cases}$$

Then $\|\nabla u_{r_k}\|_{D,B,m_A} = 1$ and, by (4.4), $c_0 \ge r_k^D \exp\big(\alpha\, k^{D'}\big) = \exp\big((\alpha-\alpha_{D,A})\,k^{D'}\big)$. Consequently, since $k \in \mathbb{N}$ might be an arbitrary number, $\alpha \le \alpha_{D,A}$.

To prove Theorem 2.12 we use test functions from the proof of Lemma 4.1 and the following analogue to [11, Lemma 1].

Lemma 4.3. *Let Φ_1, Φ_2 be Young functions such that $\Phi_1 \prec\prec \Phi_2$ and let $\mu(\Omega) < \infty$. Suppose that there exists a normed linear space W such that $W \hookrightarrow L^{\Phi_2}(\Omega,\mu)$. Then the functional*

$$J(u) = \int_\Omega \Phi_1\big(|u(x)|\big)\,\mathrm{d}\mu(x) \tag{4.5}$$

is bounded on bounded subsets of W.

Proof. The proof can be carried similarly to [11, proof of Lemma 1], however, for reader's convenience we present it here.

Since $W \hookrightarrow L^{\Phi_2}(\Omega,\mu)$, there is a constant $K > 0$ so that $\|u\|_{\Phi_2,\Omega,\mu} \le K\|u\|_W$ for all $u \in W$. Assumption $\Phi_1 \prec \Phi_2$ implies that there is a nondecreasing function $N : [0,\infty) \to [0,\infty)$ such that

$$\Phi_1(t) \le \Phi_2\Big(\frac{t}{K\lambda}\Big) \quad \text{for all } t \ge N(\lambda).$$

Hence, for any $u \in W$ and $\lambda = \|u\|_W$,

$$\int_\Omega \Phi_1\big(|u(x)|\big)\,\mathrm{d}\mu(x) \le \int_\Omega \Phi_1\big(N(\lambda)\big)\,\mathrm{d}\mu(x) + \int_\Omega \Phi_2\Big(\frac{|u(x)|}{K\lambda}\Big)\,\mathrm{d}\mu(x) \le \mu(\Omega)\Phi_1\big(N(\lambda)\big)+1,$$

and the assertion is verified. $\square$

Proof of Theorem 2.12. We restrict ourselves only to the weighted case, that is, $A_i > 0$ for some $i \in \{1,\dots,n\}$.

Statement (i). Since $0 \in \Omega$, $B(0,R) \subset \Omega$ for some $R > 0$. We can assume that $R = 1$, that is, $B \subset \Omega$ (otherwise we use a scaling argument) and consider functions u_r, $r \in (0,1)$, from (4.3) extended by zero outside B. From the proof of Lemma 4.1 we have, with a fixed $K > \alpha_{D,A}$, that

$$\|\nabla u_r\|_{D,\Omega,m_A} = 1 \quad \text{and} \quad \lim_{r\to 0_+} \int_B \exp\big(K\,|u_r(x)|^{D'}\big)x^A\,\mathrm{d}x = \infty. \tag{4.6}$$

Namely, it follows from the estimate

$$\frac{1}{m_A(B)} \int_B \exp\big(K\,|u_r(x)|^{D'}\big)x^A\,\mathrm{d}x \geq r^D \exp\Big(\frac{KD}{\alpha_{D,A}}\log(r^{-1})\Big) = r^D r^{-KD/\alpha_{D,A}}.$$

Putting

$$\Phi_1(t) = \exp(Kt^{D'}) - 1,$$

we find that the functional J

$$J(u) = \int_\Omega \Phi_1\big(|u(x)|\big)x^A\,\mathrm{d}x$$

(cf. (4.5)) is unbounded on the unit ball in the space $W = W_0^{1,D}(\Omega,m_A)$. Since, by Remark 3.5, $\Phi_1 \approx \Phi$ we have, due to Theorem 3.6, $L^{\Phi_1}(\Omega,m_A) = L^{\Phi}(\Omega,m_A)$. From Lemma 4.3 we see that, if the space $W_0^{1,D}(\Omega,m_A)$ were embedded to some Orlicz space $L^{\Psi}(\Omega,m_A)$ such that $L^{\Psi}(\Omega,m_A) \subsetneqq L^{\Phi}(\Omega,m_A)$, that is, $\Phi \prec\prec \Psi$, then J should be bounded on the unit ball in W, and it would lead to contradiction.

Statement (ii). The proof is analogous to that of [7, proof of Theorem 3.1], we are presenting it here for the reader's convenience.

Putting $v_r = K^{1/D'}u_r$, we obtain from (4.6) that

$$\|\nabla v_r\|_{D,\Omega,m_A} = K^{1/D'} \quad \text{and} \quad \lim_{r\to 0_+} \int_B \big(\exp\big(|v_r(x)|^{D'}\big) - 1\big)x^A\,\mathrm{d}x = \infty.$$

It implies that the functional

$$F(w) = \int_B \big(\exp\big(|w(x)|^{D'}\big) - 1\big)x^A\,\mathrm{d}x$$

is unbounded on the bounded set

$$\mathscr{B} = \{w \in W_0^{1,D}(\Omega,x^A)\,;\ \|\nabla w\|_{D,\Omega,m_A} \leq K^{1/D'}\}$$

of the Sobolev space $W_0^{1,D}(\Omega,m_A)$. Thus, we can find a sequence of function $\{w_j\}_{j=1}^\infty$ such that

$$F(w_j) > j, \quad j \in \mathbb{N}. \tag{4.7}$$

Assume, that embedding (2.14) is compact. Then there exist a subsequence $\{w_{j_k}\}_k \subset \{w_j\}_j$ and a function $w \in L^{\Phi}(\Omega,m_A)$ such that

$$\|w_{j_k} - w\|_{\Phi,\Omega,m_A} \to 0 \quad \text{as } k \to \infty. \tag{4.8}$$

Hence, the function w belongs to $E^{\Phi}(\Omega, m_A)$ (the closure in $L^{\Phi}(\Omega, m_A)$ of the set of all measurable functions bounded on $\overline{\Omega}$). Since $E^{\Phi}(\Omega, m_A)$ is a linear space (cf. [1, Lemma 8.15]), $2w \in E^{\Phi}(\Omega, m_A)$ and

$$F(2w) < \infty.$$

Furthermore, (4.8) implies

$$F\big(2(w_{j_k} - w)\big) \to 0 \quad \text{as } k \to \infty$$

(cf. [1, paragraph 8.13]). Then, by convexity of Φ, we obtain

$$F(w_{j_k}) = F\Big(\frac{2(w_{j_k} - w) + 2w}{2}\Big) \le \tfrac{1}{2}\big(F(2(w_{j_k} - w)) + F(2w)\big) < \infty,$$

which contradicts (4.7). □

5 Proof of Theorem 2.13 on the concentrated compactness

Throughout this section let us assume that

(a) Ω is a bounded domain in $\mathbb{R}^n$,
(b) $0 \in \Omega$ if $A_i > 0$ for some $i \in \{1, \dots, n\}$.

First observe that given a sequence $\{u_k\}_{k=1}^{\infty}$ in $W_0^{1,D}(\Omega, m_A)$, $\int_\Omega |\nabla u_k(x)|^D x^A \,\mathrm{d}x \le 1$, we can select a subsequence which satisfies conditions (2.17)

- Given a sequence $\{u_k\}_{k=1}^{\infty}$ in $W_0^{1,D}(\Omega, m_A)$, $\int_\Omega |\nabla u_k(x)|^D x^A \,\mathrm{d}x \le 1$, it has a weakly convergent subsequence in $W_0^{1,D}(\Omega, m_A)$. It means (if the subsequence is denoted again $\{u_k\}_{k=1}^{\infty}$) that there exists $u \in W_0^{1,D}(\Omega, m_A)$ such that
$$u_k \rightharpoonup u \quad \text{in } W_0^{1,D}(\Omega, m_A). \tag{5.1}$$
- Since the space $W_0^{1,D}(\Omega, m_A)$ is compactly embedded into $L^D(\Omega, m_A)$ (see (2.16)) we can select a subsequence of $\{u_k\}_{k=1}^{\infty}$, again denoted as $\{u_k\}_{k=1}^{\infty}$, satisfying
$$u_k \to u \quad \text{a.e. in } \Omega. \tag{5.2}$$
- Finally, by the duality $C\big(\overline{\Omega}\big)$, $\mathscr{M}\big(\overline{\Omega}\big)$ (see Proposition 2.6), we can select another subsequence (denoted again $\{u_k\}_{k=1}^{\infty}$) such that the sequence $|\nabla u_k(x)|^D x^A$ is weakly*-convergent to a Radon measure ν in $\mathscr{M}\big(\overline{\Omega}\big)$, that is,
$$|\nabla u_k(x)|^D x^A \overset{*}{\rightharpoonup} \nu \quad \text{in } \mathscr{M}(\overline{\Omega}). \tag{5.3}$$

To prove the Theorem 2.13 we need some preliminary work.

Lemma 5.1. *Let $\{u_k\}_{k=1}^{\infty} \subset W_0^{1,D}(\Omega, m_A)$ be such that $\int_\Omega |\nabla u_k(x)|^D x^A \,\mathrm{d}x \le 1$ and let conditions* (5.1)–(5.3) *be satisfied. Assume that $F, N \subset \overline{\Omega}$ be two disjoint compact sets and $\nu(N) > 0$. Then there exist constants $\delta > 0$ and $C > 0$ such that, for all $k \in \mathbb{N}$,*

$$\int_F \exp\big((1+\delta)\,\alpha_{D,A}\,|u_k(x)|^{D'}\big) x^A \,\mathrm{d}x \le C. \tag{5.4}$$

Proof. By (5.3), taking the test function $\varphi \equiv 1$, we obtain

$$\nu\big(\overline{\Omega}\big) \le 1. \tag{5.5}$$

Set

$$\eta = \operatorname{dist}(F, N). \tag{5.6}$$

Since F, N are compact and disjoint, $\eta > 0$. Consider the set

$$F_\eta = \big\{x \in \mathbb{R}^n\,;\ \operatorname{dist}(x, F) < \eta\big\}$$

and take a function $\psi_\eta \in C_0^\infty(\mathbb{R}^n)$ such that

$$0 \le \psi_\eta \le 1, \quad \psi_\eta = 1 \quad \text{on } F_{\eta/4}, \quad \psi_\eta = 0 \quad \text{on } \mathbb{R}^n \setminus F_{\eta/2}.$$

Hence, using (5.3), (5.5) and the assumption $\nu(N) > 0$ we have

$$\begin{aligned}\int_F |\nabla u_k(x)|^D x^A \,\mathrm{d}x \le \int_{F_{\frac{\eta}{2}} \cap \overline{\Omega}} \psi_\eta(x) |\nabla u_k(x)|^D x^A \,\mathrm{d}x \xrightarrow{k\to\infty} \int_{F_{\frac{\eta}{2}} \cap \overline{\Omega}} \psi_\eta(x) \,\mathrm{d}\nu(x) \\ \le \nu\big(\overline{\Omega}\big) - \nu(N) \le 1 - \nu(N) = \frac{1}{(1+\delta)^D}\end{aligned}$$

on putting $\delta = 1/(1-\nu(N))^{1/D} > 0$. Thus,

$$\int_F |(1+\delta)\nabla u_k(x)|^D x^A \,\mathrm{d}x \le 1, \quad k \in \mathbb{N},$$

and, applying Theorem 2.7 with $(1+\delta)u_k$ in place of u_k and F in place of Ω, we obtain (5.4). □

We need some auxiliary results. Recall that (cf. [2, Definition 5.2.2]) a bounded set $\mathscr{F} \subset L^1(G)$ (G being a domain in $\mathbb{R}^n$ such that $|G| < \infty$) is called *equi-integrable*, if given $\varepsilon > 0$ there is $\delta > 0$ such that, for every set $E \subset G$, $|G| < \delta$, then $\sup_{f\in\mathscr{F}} \int_E |f(x)| \,\mathrm{d}x < \varepsilon$.

Lemma 5.2 ([2, Lemma 5.2.5])**.** *Let G be a domain in $\mathbb{R}^n$ such that $|G| < \infty$. Suppose that a sequence $\{f_k\}_{k=1}^\infty$ is bounded in $L^1(G)$. Then the sequence $\{f_k\}_{k=1}^\infty$ is equi-integrable if and only if*

$$\lim_{b\to\infty} \sup_{k\in\mathbb{N}} \int_{\{x\in G\,;\ |f_k(x)|>b\}} |f(x)| \,\mathrm{d}x = 0.$$

We need a suitable version of the *Vitali convergence theorem*.

Lemma 5.3 ([2, Lemma 5.2.6])**.** *Let G be a domain in $\mathbb{R}^n$ such that $|G| < \infty$. Suppose that a sequence $\{f_k\}_{k=1}^\infty$ is equi-integrable in $L^1(G)$ converges a.e. to some $f \in L^1(G)$. Then*

$$\lim_{k\to\infty} \int_G f_k(x) \,\mathrm{d}x = \int_G f(x) \,\mathrm{d}x.$$

Lemma 5.4. *Let u_k, $k \in \mathbb{N}$, and u be measurable functions such that $u_k \to u$ a.e. on Ω. Let there are positive constants α, δ and C such that*

$$\int_\Omega \exp\big((1+\delta)\,\alpha\,|u_k(x)|^{D'}\big) x^A \,\mathrm{d}x \le C \quad \textit{for all } k \in \mathbb{N}. \tag{5.7}$$

Then

$$\lim_{k\to\infty} \int_\Omega \exp\big(\alpha\,|u_k(x)|^{D'}\big) x^A \,\mathrm{d}x = \int_\Omega \exp\big(\alpha\,|u(x)|^{D'}\big) x^A \,\mathrm{d}x. \tag{5.8}$$

Proof. For $\beta > 0$ and any $k \in \mathbb{N}$, condition (5.7) implies

$$\int_{\{x\in\Omega\,;\,|u_k(x)|>\beta\}} \exp\left(\alpha\,|u_k(x)|^{D'}\right) x^A \,\mathrm{d}x$$
$$= \int_{\{x\in\Omega\,;\,|u_k(x)|>\beta\}} \frac{\exp\left((1+\delta)\,\alpha\,|u_k(x)|^{D'}\right)}{\exp\left(\delta\,\alpha\,|u_k(x)|^{D'}\right)} x^A \,\mathrm{d}x \le \frac{C}{\exp\left(\delta\,\alpha\,\beta^{D'}\right)}.$$

Thus, the sequence $\{F_k\}_{k=1}^{\infty}$ of functions

$$F_k(x) = \exp\left(\alpha\,|u_k(x)|^{D'}\right) x^A, \quad k \in \mathbb{N},$$

(bounded in $L^1(\Omega)$) is, due to Lemma 5.2, equi-integrable in $L^1(\Omega)$. To complete the proof of (5.8) we apply Lemma 5.3. $\square$

Lemma 5.5. *Let* $\{u_k\}_{k=1}^{\infty} \subset W_0^{1,D}(\Omega, m_A)$ *be such that* $\int_\Omega |\nabla u_k(x)|^D x^A \,\mathrm{d}x \le 1$ *and let conditions* (5.1)–(5.3) *be satisfied. Moreover, assume that if* $u = 0$*, then* ν *is not a Dirac mass concentrated at one point. Then there exist constants* $C > 0$ *and* $p > 1$ *such that* (2.18) *holds for all* $k \in \mathbb{N}$.

Proof. The assumption on ν implies that there exists a compact set $N_1 \subset \overline{\Omega}$ such that $0 < \nu(N_1) < \nu\left(\overline{\Omega}\right) \le 1$ (cf. (5.5)). Define the following subsets of $\overline{\Omega}$:

$$G = \overline{\Omega} \setminus N_1 \quad \text{and} \quad G_\tau = \left\{x \in \overline{\Omega}\,;\, \mathrm{dist}(x, N_1) > \tau\right\},\ \tau > 0.$$

By the regularity of the Radon measure ν we have

$$\lim_{\tau\to 0_+} \nu(G_\tau) = \nu(G) = \nu\left(\overline{\Omega}\right) - \nu(N_1) \in \left(0, \nu\left(\overline{\Omega}\right)\right).$$

Thus, there is $\tau > 0$ such that

$$0 < \nu(G_{2\tau}) \le \nu(G_\tau) < \nu\left(\overline{\Omega}\right).$$

Observe that $F_1 = \overline{G_\tau}$ is compact and $F_1 \cap N_1 = \emptyset$. Applying Lemma 5.1 with $F = F_1$ and $N = N_1$ we find constants $\delta_1 > 0$ and $C_1 > 0$ such that, for all $k \in \mathbb{N}$,

$$\int_{F_1} \exp\left((1+\delta_1)\,\alpha_{D,A}\,|u_k(x)|^{D'}\right) x^A \,\mathrm{d}x \le C_1. \tag{5.9}$$

Put $N_2 = \overline{G_{2\tau}}$ and $F_2 = \overline{\Omega} \setminus G_\tau$. Obviously these sets are compact and disjoint. Moreover,

$$\nu(N_2) \ge \nu(G_{2\tau}) > 0.$$

Therefore we can apply Lemma 5.1 again, now with $F = F_2$ and $N = N_2$, and find constants $\delta_2 > 0$ and $C_2 > 0$ such that, for all $k \in \mathbb{N}$,

$$\int_{F_2} \exp\left((1+\delta_2)\,\alpha_{D,A}\,|u_k(x)|^{D'}\right) x^A \,\mathrm{d}x \le C_2. \tag{5.10}$$

To complete the proof we use (5.9), (5.10) and the fact that $F_1 \cup F_2 = \overline{\Omega}$ to obtain (2.18) with $p = \min\{1+\delta_1, 1+\delta_2\}$ and $C = C_1 + C_2$. $\square$

Lemma 5.6. *Let $\{u_k\}_{k=1}^{\infty} \subset W_0^{1,D}(\Omega, m_A)$ be such that $\int_\Omega |\nabla u_k(x)|^D x^A \,\mathrm{d}x \le 1$ Assume that conditions* (5.1),(5.2) *with $u = 0$, and condition*

$$|\nabla u_k(x)|^D x^A \overset{*}{\rightharpoonup} \delta_{x_0} \quad \textit{(where } x_0 \in \overline{\Omega}\textit{) in } \mathscr{M}(\overline{\Omega}) \tag{5.11}$$

are satisfied. Then the following properties hold.

(i) *If there is a constant $c \ge 0$ such that*

$$\lim_{k\to\infty} \int_\Omega \big(\exp\big(\alpha_{D,A}\, |u_k(x)|^{D'}\big) - 1\big) x^A \,\mathrm{d}x = c,$$

then

$$\big(\exp\big(\alpha_{D,A}\, |u_k(x)|^{D'}\big) - 1\big) x^A \overset{*}{\rightharpoonup} c\, \delta_{x_0}(x) \quad \textit{in } \mathscr{M}(\overline{\Omega}).$$

(ii) *The sequence $\{f_k\}_{k=1}^{\infty}$ of functions*

$$f_k(x) = \big(\exp(\alpha_{D,A}\, |u_k(x)|^{D'}) - 1\big) x^A, \quad k \in \mathbb{N},$$

is relatively compact with respect to the weak*-convergence in $\mathscr{M}(\overline{\Omega})$ and the limits of convergent subsequences belong to the set

$$\big\{c\, \delta_{x_0};\ 0 \le c \le (S-1) m_A(\Omega)\big\},$$

where

$$S = \sup_u \frac{1}{m_A(\Omega)} \int_\Omega \exp\big(\alpha_{D,A}\, |u(x)|^{D'}\big) x^A \,\mathrm{d}x \tag{5.12}$$

through all $u \in W_0^{1,D}(\Omega, m_A)$ satisfying $\int_\Omega |\nabla u(x)|^D x^A \,\mathrm{d}x \le 1$.

Proof. Statement (i). At first observe that, for any $\eta > 0$,

$$\lim_{k\to\infty} \int_{\Omega \setminus B(x_0,\eta)} \big(\exp\big(\alpha_{D,A}\, |u_k(x)|^{D'}\big) - 1\big) x^A \,\mathrm{d}x = 0. \tag{5.13}$$

To see this, take $N = \overline{B(x_0, \eta/2)}$ in Lemma 5.1 and find positive constants δ and C such that $\int_{\Omega \setminus B(x_0,\eta)} \exp\big((1+\delta)\, \alpha_{D,A}\, |u_k(x)|^D\big) x^A \,\mathrm{d}x \le C$. Then use Lemma 5.4 and the fact that $u_k \to 0$ a.e. on Ω.

Thus, for any $\eta > 0$,

$$\lim_{k\to\infty} \int_{B(x_0,\eta)} \big(\exp\big(\alpha_{D,A}\, |u_k(x)|^{D'}\big) - 1\big) x^A \,\mathrm{d}x = c. \tag{5.14}$$

For an arbitrary test function $\psi \in C\big(\overline{\Omega}\big)$ and $\varepsilon > 0$ find $\eta > 0$ such that, for any $x \in \overline{\Omega}$,

$$|x - x_0| < \eta \quad \Longrightarrow \quad |\psi(x) - \psi(x_0)| < \frac{\varepsilon}{6 \max\{1, c\}}. \tag{5.15}$$

We obtain the following estimate

$$
\begin{aligned}
\mathscr{I} = \Big| \int_\Omega \psi(x)\, \mathrm{d}(c\,\delta_{x_0}(x)) - \int_\Omega \psi(x)\big(\exp\big(\alpha_{D,A}\,|u_k(x)|^{D'}\big)-1\big)x^A\,\mathrm{d}x\Big| \\
= \Big| c\,\psi(x_0) - \int_\Omega \psi(x)\big(\exp\big(\alpha_{D,A}\,|u_k(x)|^{D'}\big)-1\big)x^A\,\mathrm{d}x\Big| \\
\le \int_{\Omega\setminus B(x_0,\eta)} |\psi(x)|\big(\exp\big(\alpha_{D,A}\,|u_k(x)|^{D'}\big)-1\big)x^A\,\mathrm{d}x \\
+ \int_{B(x_0,\eta)} |\psi(x)-\psi(x_0)|\big(\exp\big(\alpha_{D,A}\,|u_k(x)|^{D'}\big)-1\big)x^A\,\mathrm{d}x \\
+ |\psi(x_0)|\Big| c - \int_{B(x_0,\eta)} \big(\exp\big(\alpha_{D,A}\,|u_k(x)|^{D'}\big)-1\big)x^A\,\mathrm{d}x\Big| = \mathscr{I}_1+\mathscr{I}_2+\mathscr{I}_3.
\end{aligned}
$$

For the estimate of the first term we use the fact that ψ is bounded on $\overline{\Omega}$ and (5.13) to find $k_1 \in \mathbb{N}$ such that, for all $k \ge k_1$, $\mathscr{I}_1 \le \varepsilon/3$. For the second term we use (5.14) to find $k_2 \in \mathbb{N}$ such that, for all $k \ge k_2$,

$$\int_{B(x_0,\eta)} \big(\exp\big(\alpha_{D,A}\,|u_k(x)|^{D'}\big)-1\big)x^A\,\mathrm{d}x \le 2c.$$

Hence, by the estimate (5.15),

$$\mathscr{I}_2 = \int_{B(x_0,\eta)} |\psi(x)-\psi(x_0)|\big(\exp\big(\alpha_{D,A}\,|u_k(x)|^{D'}\big)-1\big)x^A\,\mathrm{d}x \le \frac{\varepsilon}{6\max\{1,c\}}2c \le \frac{\varepsilon}{3}.$$

Finally, by (5.14) we find $k_3 \in \mathbb{N}$ such that, for all $k \ge k_3$, $\mathscr{I}_3 < \frac{\varepsilon}{3}$. Thus, for all $k \ge \max\{k_1,k_2,k_3\}$, $\mathscr{I} \le \mathscr{I}_1+\mathscr{I}_2+\mathscr{I}_3 < \varepsilon$, which completes the proof of the first statement.

Statement (ii). The inequality $0 \le c \le (S-1)m_A(\Omega)$ follows from Theorem 4.1, that is, from the fact that the supremum S in (5.12) is finite. Since the sequence $\{f_k\}_{k=1}^\infty$ is bounded in $L^1(\Omega)$, it is relatively compact with respect to the weak*-convergence in $\mathscr{M}(\overline{\Omega})$. Thus, there exists a subsequence $\{f_{k_i}\}_{i=1}^\infty \subset \{f_k\}_{k=1}^\infty$ such that $f_{k_i} \overset{*}{\rightharpoonup} \gamma$ in $\mathscr{M}(\overline{\Omega})$. That means, taking $\psi \equiv 1$ on $\overline{\Omega}$ for the test function, that

$$
\begin{aligned}
c\,\delta_{x_0} = c = \lim_{i\to\infty} \int_\Omega \big(\exp\big(\alpha_{D,A}\,|u_{k_i}(x)|^{D'}\big)-1\big)x^A\,\mathrm{d}x \\
= \lim_{i\to\infty} \int_\Omega \psi(x)\big(\exp\big(\alpha_{D,A}\,|u_{k_i}(x)|^{D'}\big)-1\big)x^A\,\mathrm{d}x = \int_\Omega \psi(x)\,\mathrm{d}\gamma(x) = \gamma(\Omega),
\end{aligned}
$$

which completes the proof. □

Lemma 5.7. *Let* $\{u_k\}_{k=1}^\infty \subset W_0^{1,D}(\Omega,m_A)$ *be such that* $\int_\Omega |\nabla u_k(x)|^D x^A\,\mathrm{d}x \le 1$ *Suppose that conditions* (5.1) *and* (5.2) *hold and that* $u \ne 0$. *Then there exist constants* $C>0$ *and* $\delta>0$ *such that, for all* $k \in \mathbb{N}$,

$$\int_\Omega \exp\big((1+\delta)\,\alpha_{D,A}\,|u_k(x)|^{D'}\big)x^A\,\mathrm{d}x \le C. \tag{5.16}$$

To prove the lemma we need the following result.

Lemma 5.8. *Let $R>0$ and let $\{g_k\}_{k=1}^{\infty}$ be a sequence of non-increasing absolutely continuous functions on the interval $[0,R]$ such that $g_k(R)=0$. Set $u_k(x)=g_k(|x|)$ for $x\in B(0,R)$ and $k\in\mathbb{N}$ and suppose that $\int_{B(0,R)}|\nabla u_k(x)|^D x^A\,\mathrm{d}x\le 1$, $k\in\mathbb{N}$. In addition, assume that (5.1)–(5.3) hold. If, for any $\delta>0$,*

$$\lim_{k\to\infty}\int_{B(0,R)}\exp\big((1+\delta)\,\alpha_{D,A}\,|u_k(x)|^{D'}\big)x^A\,\mathrm{d}x=\infty, \tag{5.17}$$

then, for each $r\in(0,R)$,

$$\lim_{k\to\infty}u_k=0\quad \textit{uniformly on } B(0,R)\setminus B(0,r). \tag{5.18}$$

Proof. Observe that, for any $r\in(0,R)$,

$$\lim_{k\to\infty}\int_{B(0,R)\setminus B(0,r)}|\nabla u_k(x)|^D x^A\,\mathrm{d}x=0. \tag{5.19}$$

Indeed, if it is not the case, then (possibly passing to a subsequence) we find $\varepsilon\in(0,1)$ and $r_0\in(0,R)$ such that, for all $k\in\mathbb{N}$, $\int_{B(0,R)\setminus B(0,r_0)}|\nabla u_k(x)|^D x^A\,\mathrm{d}x\ge\varepsilon$. Thus, taking a test function $\psi\in C\big(\overline{B(0,R)}\big)$ such that $0\le\psi\le 1$ and $\psi\equiv 1$ on $B(0,R)\setminus B(0,r_0)$, we obtain that

$$\nu\big(B(0,R)\setminus B(0,r_0)\big)>0.$$

However, by Lemma 5.1, it leads to a contradiction and so (5.19) holds.

To prove (5.18) we use the fact that the functions u_k, $k\in\mathbb{N}$, are radially non-increasing. Therefore, if $x\in B(0,R)\setminus B(0,r_0)$, then $r\le|x|<R$, and so

$$0\le u_k(x)=g_k(|x|)\le g_k(r),\quad k\in\mathbb{N}.$$

Thus, it is enough to show that

$$\lim_{k\to\infty}g_k(r)\to 0. \tag{5.20}$$

Since the functions g_k, $k\in\mathbb{N}$, are absolutely continuous and satisfy $g_k(R)=0$, we have $g_k(r)=-\int_r^R g_k'(t)\,\mathrm{d}t$. Then, using the Hölder inequality (cf. also (4.2)) and (5.19), we obtain

$$\begin{aligned}0\le g_k(r)&=\int_r^R|g_k'(t)|\,t^{\frac{D-1}{D}}t^{-\frac{D-1}{D}}\,\mathrm{d}t\le\Big(\int_r^R|g_k'(t)|^D t^{D-1}\Big)^{1/D}\Big(\int_r^R t^{-1}\,\mathrm{d}t\Big)^{1/D'}\\&=\big(\log\tfrac{R}{r}\big)^{1/D'}\Big(\int_{\partial B}x^A\,\mathrm{d}\sigma(x)\Big)^{-D}\Big(\int_{B(0,R)\setminus B(0,r)}|\nabla u_k(x)|^D x^A\,\mathrm{d}x\Big)^{1/D}\to 0\quad\text{when } k\to\infty,\end{aligned}$$

which verifies (5.20). □

Further we use a result of Cabré and Ros-Oton [4, Proposition 4.2] which is a direct consequence of their isoperimetric inequality ([4, inequality (1.7)]) and a result of Talenti [24]. The following proposition is also mentioned in [12, Lemma 2.1].

Lemma 5.9. *Let u be a Lipschitz continuous function in $\mathbb{R}^n$ such that $m_A\big(\{x\in\mathbb{R}^n\,;\,|u(x)|>t\}\big)<\infty$ for every $t>0$. Then there exists a radial rearrangement $u^\star$ of u such that*

(i) *$u^\star$ is nonnegative and radially decreasing;*
(ii) *$m_A\big(\{x\in\mathbb{R}^n\,;\,|u(x)|>t\}\big)=m_A\big(\{x\in\mathbb{R}^n\,;\,u^\star(x)>t\}\big)$ for all $t>0$;*

(iii) *for every Young function Φ*

$$\int_{\mathbb{R}^n} \Phi\big(|\nabla u^\bigstar(x)|\big) x^A \, \mathrm{d}x \le \int_{\mathbb{R}^n} \Phi\big(|\nabla u(x)|\big) x^A \, \mathrm{d}x;$$

(iv) *if $\Psi : (0,\infty) \to (0,\infty)$ is a nondecreasing and measurable function, then*

$$\int_{\mathbb{R}^n} \Psi\big(u^\bigstar(x)\big) x^A \, \mathrm{d}x = \int_{\mathbb{R}^n} \Psi\big(|u(x)|\big) x^A \, \mathrm{d}x.$$

This symmetrization we use for proving Lemma 5.7.

Proof of Lemma 5.7. We show (5.16) by contradiction. If it is not the case, using density of $C_0^\infty(\Omega)$ in $W_0^{1,D}(\Omega, m_A)$, we can find a sequence of functions $v_k \in C_0^\infty(\Omega)$, $k \in \mathbb{N}$, such that

$$\int_\Omega |\nabla v_k(x)|^D x^A \, \mathrm{d}x \le 1, \quad v_k \rightharpoonup u \text{ in } W_0^{1,D}(\Omega, m_A)$$

and

$$\lim_{k\to\infty} \int_\Omega \exp\big((1+\delta)\, \alpha_{D,A}\, |v_k(x)|^{D'}\big) x^A \, \mathrm{d}x = \infty \quad \text{for every } \delta > 0. \tag{5.21}$$

By Lemma 5.9 we obtain (possibly passing to a subsequence) that

$$\int_{B(0,R)} |\nabla v_k^\bigstar(x)|^D x^A \, \mathrm{d}x \le 1 \quad \text{and} \quad v_k^\bigstar \rightharpoonup u^\bigstar \text{ in } W_0^{1,D}(B(0,R), m_A),$$

moreover,

$$v_k^\bigstar \to u^\bigstar \quad \text{in } L^D(B(0,R), m_A) \quad \text{and} \quad u^\bigstar \ne 0.$$

Further, by Lemma 5.9 (iv),

$$\lim_{k\to\infty} \int_{B(0,R)} \exp\big((1+\delta)\, \alpha_{D,A}\, v_k^\bigstar(x)^{D'}\big) x^A \, \mathrm{d}x = \infty \quad \text{for every } \delta > 0.$$

Applying Lemma 5.8 to the sequence $\{v_k^\bigstar\}_{k=1}^\infty$ (possibly passing to a subsequence) we find that, for each $r \in (0,R)$,

$$\lim_{k\to\infty} v_k^\bigstar = 0 \quad \text{uniformly on } B(0,R) \setminus B(0,r).$$

However, it contradicts to $u^\bigstar \ne 0$. □

Proof of Theorem 2.13. *Statement* (i). It follows from Lemma 5.6.
Statement (ii). The assertion directly follows from Lemma 5.5.
Statement (iii). It is a consequence of Lemma 5.7.
The assertion (2.19) then follows, due to (5.2), by Lemma 5.4. □

6 Epilogue

Comparison with a result of Lam We want to point out that our Theorem 2.7 is slightly different from the result of Lam [12, Theorem 1.1]. The main difference is that Lam considers the domain

$$\Omega^* = \big\{(x_1, \ldots, x_n) \in \Omega \,;\, x_i > 0 \quad \text{whenever} \quad A_i > 0\big\}$$

instead of Ω. Consequently, his optimal constant $\alpha_{D,A}^*$ is different from our $\alpha_{D,A}$. Namely,

$$\alpha_{D,A}^* = 2^{-k/(D-1)} \alpha_{D,A},$$

where k is the number of strictly positive entries of $A = (A_1, \ldots, A_n)$. It is not difficult to derive all the corresponding results for the setting of Lam.

Existence of an extremal function Let us mention an interesting question about the existence of an *extremal function* in (2.12) with $\alpha = \alpha_{D,A}$. That means, we are interested if there exists a function from the Sobolev space $W_0^{1,D}(\Omega, m_A)$ for which the supremum

$$\sup_{\|\nabla u\|_{D,\Omega,m_A} \le 1} \frac{1}{m_A(\Omega)} \int_\Omega \exp\left(\alpha_{D,A} |u(x)|^{D'}\right) x^A \, \mathrm{d}x$$

is attained. The answer is positive for the classical case $A_1 = \cdots = A_n = 0$, $D = n$. The first result in this direction is due to L. Carleson and A. Chang [5], who proved the existence of an extremal function when Ω is a ball in $\mathbb{R}^n$. It was extended to arbitrary bounded domain Ω in $\mathbb{R}^2$ by M. Flutcher in [9] and to arbitrary bounded domain Ω in $\mathbb{R}^n$ by K. Lin in [13]. More information can be found also in the papers [14] and [22]. To keep this paper readable we decided to include our results concerning this topic in our forthcoming papers. At this moment we are not aware of any results in the case when at least one of the numbers $A_1, \dots, A_n$ is nonzero.

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