# On Segal–Sugawara vectors for the orthogonal and symplectic Lie algebras

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**Abstract**

We consider the center of the affine vertex algebra at the critical level associated with the simple Lie algebras of types $B$, $C$ and $D$. We derive new formulas for the generators of the center which were produced earlier with the use of the Brauer algebra symmetrizer.

# 1 Introduction

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ equipped with a standard symmetric invariant bilinear form. The affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ is defined as the central extension

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K \tag{1.1}$$

of the Lie algebra of Laurent polynomials in $t$. The vacuum module $V_{\text{cri}}(\mathfrak{g})$ at the critical level over $\widehat{\mathfrak{g}}$ is the quotient of the universal enveloping algebra $\mathrm{U}(\widehat{\mathfrak{g}})$ by the left ideal generated by $\mathfrak{g}[t]$ and $K + h^\vee$. The vacuum module has a vertex algebra structure and is known as the (*universal*) *affine vertex algebra*; see e.g. [4] and [6]. The *center* of the vertex algebra $V_{\text{cri}}(\mathfrak{g})$ is defined by

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \{S \in V_{\text{cri}}(\mathfrak{g}) \mid \mathfrak{g}[t]\, S = 0\}.$$

Any element of $\mathfrak{z}(\widehat{\mathfrak{g}})$ is called a *Segal–Sugawara vector*. The vertex algebra axioms imply that the center is a commutative associative algebra which can be regarded as a subalgebra of $\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$. The algebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ is equipped with the derivation $T = -d/dt$ arising from the vertex algebra structure. By a theorem of Feigin and Frenkel [3], the differential algebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ possesses generators $S_1, \dots, S_n$ so that $\mathfrak{z}(\widehat{\mathfrak{g}})$ is the algebra of polynomials

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathbb{C}[T^r S_l \mid l = 1, \dots, n, \ \ r \geqslant 0],$$

where $n = \operatorname{rank} \mathfrak{g}$; see also [4]. The algebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ is known as the *Feigin–Frenkel center*, and we call $S_1, \dots, S_n$ a *complete set of Segal–Sugawara vectors*. According to [3] (see also [4]), the center can be identified with the *classical $\mathcal{W}$-algebra* associated with the Langlands dual Lie algebra ${}^L\mathfrak{g}$ via an affine version of the Harish-Chandra isomorphism

$$\mathfrak{z}(\widehat{\mathfrak{g}}) \cong \mathcal{W}({}^L\mathfrak{g}). \tag{1.2}$$

Explicit formulas for complete sets of Segal–Sugawara vectors were given in [1] and [2] for the Lie algebras $\mathfrak{g}$ of type $A$ and in [7] for types $B$, $C$ and $D$ with the use of the Brauer algebra. Their images with respect to the Harish-Chandra isomorphism (1.2) were found in [9]; see also [8] for a detailed exposition of these results and applications to commutative subalgebras in enveloping algebras and to higher order Hamiltonians in the Gaudin models. Complete sets of Segal–Sugawara vectors for the Lie algebra of type $G_2$ were produced in [10] by using computer-assisted calculations. A different way to construct generators of $\mathfrak{z}(\widehat{\mathfrak{g}})$ was developed in [11] which lead to new explicit formulas for the Lie algebras of types $B, C, D$ and $G_2$.

In this note we derive new formulas for the Segal–Sugawara vectors produced in [7] for types $B$, $C$ and $D$ by eliminating the dependence on the Brauer diagrams with horizontal edges. In particular, the vectors are given explicitly in the symplectic case thus resolving the 'analytic continuation' procedure used in [7]; see also [8, Ch. 8]. We also show that the Segal–Sugawara vectors in this case coincide with those in [11] obtained by a different method. Such an identification in the orthogonal case is not immediately clear.

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# 2 Segal–Sugawara vectors

## 2.1 Definitions and notation

We will regard the orthogonal Lie algebra $\mathfrak{o}_N$ with $N = 2n + 1$ and $N = 2n$ and symplectic Lie algebra $\mathfrak{sp}_N$ with $N = 2n$ as subalgebras of $\mathfrak{gl}_N$ spanned by the elements $F_{ij}$,

$$F_{ij} = E_{ij} - E_{j'i'} \qquad \text{and} \qquad F_{ij} = E_{ij} - \varepsilon_i \, \varepsilon_j \, E_{j'i'},$$

respectively, for $\mathfrak{o}_N$ and $\mathfrak{sp}_N$, where the $E_{ij}$ are the standard basis elements of the Lie algebra $\mathfrak{gl}_N$ and $i' = N - i + 1$. In the symplectic case we set $\varepsilon_i = 1$ for $i = 1, \dots, n$ and $\varepsilon_i = -1$ for $i = n + 1, \dots, 2n$.

We will use the notation $F_{ij}[r] = F_{ij} t^r$ with $r \in \mathbb{Z}$ for elements of the Kac–Moody algebra $\widehat{\mathfrak{g}}$ for $\mathfrak{g} = \mathfrak{o}_N$ or $\mathfrak{sp}_N$, as defined in (1.1). We will regard the $N \times N$ matrix $F[r] = \big[F_{ij}[r]\big]$ as the element

$$F[r] = \sum_{i,j=1}^{N} e_{ij} \otimes F_{ij}[r] \in \operatorname{End} \mathbb{C}^N \otimes \mathrm{U}(\widehat{\mathfrak{g}}),$$

where $e_{ij}$ are the standard matrix units. It has the skew-symmetry property $F[r] + F[r]^{\mathrm{t}} = 0$ with respect to the transposition defined by

$$\mathrm{t} : e_{ij} \mapsto \begin{cases} e_{j'i'} & \text{in the orthogonal case,} \\ \varepsilon_i \varepsilon_j \, e_{j'i'} & \text{in the symplectic case.} \end{cases} \tag{2.1}$$

For each $a \in \{1, \dots, m\}$ introduce the element $F[r]_a$ of the algebra

$$\underbrace{\operatorname{End} \mathbb{C}^N \otimes \ldots \otimes \operatorname{End} \mathbb{C}^N}_{m} \otimes \mathrm{U}(\widehat{\mathfrak{g}}) \tag{2.2}$$

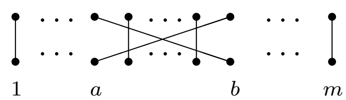
by

$$F[r]_a = \sum_{i,j=1}^{N} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(m-a)} \otimes F_{ij}[r].$$

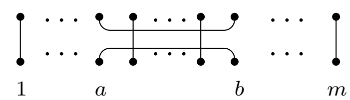
The $a$-th partial transposition $\mathrm{t}_a$ on the algebra (2.2) acts as the map (2.1) on the $a$-th copy of $\mathrm{End}\,\mathbb{C}^N$ and as the identity map on all other tensor factors.

## 2.2 Symmetrizer in the Brauer algebra

An $m$-diagram $d$ is a collection of $2m$ dots arranged into two rows with $m$ dots in each row connected by $m$ edges such that any dot belongs to only one edge. The product $dd'$ of two diagrams $d$ and $d'$ is determined by placing $d$ under $d'$ and identifying the vertices of the bottom row of $d'$ with the corresponding vertices in the top row of $d$. Let $s$ be the number of closed loops obtained in this placement. The product $dd'$ is given by $\omega^s$ times the resulting diagram without loops. The algebra $\mathcal{B}_m(\omega)$ is defined as the $\mathbb{C}(\omega)$-linear span of the $m$-diagrams with this multiplication. For $1 \leqslant a < b \leqslant m$ denote by $s_{a\,b}$ and $\epsilon_{a\,b}$ the respective diagrams of the form




and




They generate the algebra $\mathcal{B}_m(\omega)$. Its subalgebra spanned over $\mathbb{C}$ by the diagrams without horizontal edges will be identified with the group algebra of the symmetric group $\mathbb{C}[\mathfrak{S}_m]$ so that $s_{a\,b}$ is identified with the transposition $(a\,b)$.

We will use a special element $s^{(m)} \in \mathcal{B}_m(\omega)$, known as the *symmetrizer*. Several explicit expressions for $s^{(m)}$ are collected in [8, Ch. 1]; we will recall one of them, as appeared in [5],

$$s^{(m)} = \frac{1}{m!} \sum_{r=0}^{\lfloor m/2 \rfloor} (-1)^r \binom{\omega/2 + m - 2}{r}^{-1} \sum_{d \in \mathcal{D}^{(r)}} d, \tag{2.3}$$

where $\mathcal{D}^{(r)} \subset \mathcal{B}_m(\omega)$ denotes the set of diagrams which have exactly $r$ horizontal edges in the top (and hence in the bottom) row. Since $\mathcal{D}^{(0)} = \mathfrak{S}_m$, the element

$$h^{(m)} = \frac{1}{m!} \sum_{d \in \mathcal{D}^{(0)}} d \tag{2.4}$$

is the symmetrizer in $\mathbb{C}[\mathfrak{S}_m]$.

The Brauer algebra $\mathcal{B}_m(\omega)$ with the special values $\omega = N$ and $\omega = -N$ acts on the tensor space

$$(\mathbb{C}^N)^{\otimes m} = \underbrace{\mathbb{C}^N \otimes \mathbb{C}^N \otimes \ldots \otimes \mathbb{C}^N}_{m} \tag{2.5}$$

so that the action centralizers the respective actions of the orthogonal and symplectic groups. In the orthogonal case, the generators of $\mathcal{B}_m(N)$ act by the rule

$$s_{ab} \mapsto P_{ab}, \qquad \epsilon_{ab} \mapsto Q_{ab}, \qquad 1 \leqslant a < b \leqslant m, \tag{2.6}$$

where $P_{ab}$ is defined by

$$P_{ab} = \sum_{i,j=1}^{N} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{ji} \otimes 1^{\otimes(m-b)}, \tag{2.7}$$

while

$$Q_{ab} = \sum_{i,j=1}^{N} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{i'j'} \otimes 1^{\otimes(m-b)}.$$

In the symplectic case, the action of $\mathcal{B}_m(-N)$ with $N = 2n$ in the space (2.5) is defined by

$$s_{ab} \mapsto -P_{ab}, \qquad \epsilon_{ab} \mapsto -Q_{ab}, \qquad 1 \leqslant a < b \leqslant m, \tag{2.8}$$

where $P_{ab}$ is defined in (2.7), and

$$Q_{ab} = \sum_{i,j=1}^{2n} \varepsilon_i \varepsilon_j \, 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{i'j'} \otimes 1^{\otimes(m-b)}.$$

## 2.3 Formulas for the Segal–Sugawara vectors

We will denote by $S^{(m)}$ the image of the symmetrizer $s^{(m)} \in \mathcal{B}_m(\omega)$ under the respective actions (2.6) and (2.8), assuming $m \leqslant n$ in the symplectic case. Using the rational function

$$\gamma_m(\omega) = \frac{\omega + m - 2}{\omega + 2m - 2}$$

define the elements $\phi_m$ of the vacuum module $V_{\text{cri}}(\mathfrak{o}_N) \cong \mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ by

$$\phi_m = \gamma_m(\omega)\, \mathrm{tr}_{1,\dots,m}\, S^{(m)} \mathcal{F}_1 \dots \mathcal{F}_m, \tag{2.9}$$

where $\mathcal{F} = T + F[-1]$ and $\omega = N$ and $\omega = -N$, respectively, for the orthogonal and symplectic cases. The trace is taken with respect to all $m$ copies of the endomorphism algebra $\mathrm{End}\,\mathbb{C}^N$ in (2.2). In the symplectic case the values of $m$ are restricted to $1 \leqslant m \leqslant 2n+1$ with an additional justification of the formula (2.9) for $n+1 \leqslant m \leqslant 2n+1$ via an 'analytic continuation' argument; see [8, Sec. 8.3].

As proved in [7] (see also [8, Ch. 8]), all elements $\phi_m$ belong to the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ (they are denoted by $\phi_{mm}$ therein). Moreover, the elements $\phi_2, \phi_4, \dots, \phi_{2n}$ form a complete set of Segal–Sugawara vectors for $\mathfrak{g} = \mathfrak{o}_{2n+1}$ and $\mathfrak{sp}_{2n}$, whereas $\phi_2, \phi_4, \dots, \phi_{2n-2}, \mathrm{Pf}\, F[-1]$ form a complete set of Segal–Sugawara vectors for $\mathfrak{g} = \mathfrak{o}_{2n}$, where $\mathrm{Pf}\, F[-1]$ is the (noncommutative) *Pfaffian* of the matrix $F[-1]$.

Expand the product $\mathcal{F}_1 \dots \mathcal{F}_m$ in (2.9) as a linear combination of monomials of the form

$$F[-r_1]_{a_1} \dots F[-r_s]_{a_s}, \qquad 1 \leqslant a_1 < \dots < a_s \leqslant m, \tag{2.10}$$

with $r_1 + \dots + r_s = m$ and $r_i \geqslant 1$. By [8, Lemmas 8.1.5 & 8.3.1], any permutation of the factors $F[-r_i]_{a_i}$ in the expression

$$\mathrm{tr}_{1,\dots,m}\, S^{(m)} F[-r_1]_{a_1} \dots F[-r_s]_{a_s} \tag{2.11}$$

does not change its value. Therefore, applying conjugations by suitable permutations of the index set $1, \dots, m$ and using the cyclic property of trace, we can bring (2.11) to the form

$$\mathrm{tr}_{1,\dots,m}\, S^{(m)} F[-p_1]_1 \dots F[-p_s]_s,$$

where $1 \leqslant p_1 \leqslant \dots \leqslant p_s$ is the sequence $r_1, \dots, r_s$ arranged in the increasing order. This determines the components $\Phi_s^{(m)}$ in the expansion

$$\mathrm{tr}_{1,\dots,m}\, S^{(m)} \mathcal{F}_1 \dots \mathcal{F}_m = \sum_{s=1}^{m} \mathrm{tr}_{1,\dots,m}\, S^{(m)} \Phi_s^{(m)}, \tag{2.12}$$

so that $\Phi_s^{(m)}$ is a linear combination of monomials of the form $F[-p_1]_1 \dots F[-p_s]_s$. Note that the summands in (2.12) with odd values of $s$ are equal to zero because the matrices $F[-p_a]$ are skew-symmetric with respect to the transposition $\mathrm{t}$, while $S^{(m)}$ is stable under the simultaneous transpositions with respect to all $m$ copies of $\mathrm{End}\, \mathbb{C}^N$; see e.g. [8, Prop. 1.2.8].

*Example* 2.1. For $m = 4$ we have

$$\mathrm{tr}_{1,2,3,4}\, S^{(4)} \mathcal{F}_1 \mathcal{F}_2 \mathcal{F}_3 \mathcal{F}_4 = \mathrm{tr}_{1,2,3,4}\, S^{(4)} \Big( \Phi_2^{(4)} + \Phi_4^{(4)} \Big)$$

with

$$\Phi_2^{(4)} = 8\, F[-1]_1 F[-3]_2 + 3\, F[-2]_1 F[-2]_2 \quad \text{and} \quad \Phi_4^{(4)} = F[-1]_1 F[-1]_2 F[-1]_3 F[-1]_4;$$

cf. [8, Sec. 8.2]. □

To state the main theorem, denote by $H^{(m)}$ the image of the element $h^{(m)}$ defined in (2.4), under the respective actions (2.6) and (2.8) of the Brauer algebra. Note that $H^{(m)}$ acts as the symmetrization operator on the tensor space (2.5) in the orthogonal case, and as the anti-symmetrization operator in the symplectic case.

**Theorem 2.2.** *For $k = 1, \dots, n$ the Segal–Sugawara vector $\phi_{2k}$ is given by*

$$\phi_{2k} = \sum_{l=1}^{k} \binom{\omega + 2k - 2}{2k - 2l} \binom{2k}{2l}^{-1} \mathrm{tr}_{1,\dots,2l}\, H^{(2l)} \Phi_{2l}^{(2k)}. \tag{2.13}$$

*Proof.* We consider the orthogonal and symplectic cases simultaneously and assume in the calculations in the symplectic case that $2k \leqslant n$; then apply the same arguments as in [8, Sec. 8.3] to extend the result to all values of $k$. By (2.12) we have

$$\phi_{2k} = \gamma_{2k}(\omega)\,\mathrm{tr}_{1,\dots,2k}\, S^{(2k)}\mathcal{F}_1 \dots \mathcal{F}_{2k} = \gamma_{2k}(\omega) \sum_{l=1}^{k} \mathrm{tr}_{1,\dots,2k}\, S^{(2k)}\Phi^{(2k)}_{2l}.$$

Now use [8, Lemma 1.3.2] to calculate the partial traces to get

$$\gamma_{2k}(\omega)\,\mathrm{tr}_{2l+1,\dots,2k}\, S^{(2k)} = \binom{\omega+2k-2}{2k-2l}\binom{2k}{2l}^{-1} \gamma_{2l}(\omega)\, S^{(2l)}.$$

The desired formula will be implied by the relation

$$\gamma_{2l}(\omega)\,\mathrm{tr}_{1,\dots,2l}\, S^{(2l)}\Phi^{(2k)}_{2l} = \mathrm{tr}_{1,\dots,2l}\, H^{(2l)}\Phi^{(2k)}_{2l} \tag{2.14}$$

which is a consequence of the relation in the Brauer algebra provided by the following lemma. For every $a \in \{1,\dots,m\}$ introduce the transposition $\mathrm{t}_a$ as the linear map

$$\mathrm{t}_a : \mathcal{B}_m(\omega) \to \mathcal{B}_m(\omega), \qquad d \mapsto d^{\,\mathrm{t}_a},$$

where the diagram $d^{\,\mathrm{t}_a}$ is obtained from $d$ by swapping the $a$-th vertices in the top and bottom rows. In particular, $s_{ab}^{\mathrm{t}_a} = \epsilon_{ab}$ and $\epsilon_{ab}^{\mathrm{t}_a} = s_{ab}$. Denote by $\mathrm{J}_m$ the subspace of $\mathcal{B}_m(\omega)$ spanned by all sums $d + d^{\,\mathrm{t}_a}$ with $d \in \mathcal{B}_m(\omega)$ and $a = 1,\dots,m$.

**Lemma 2.3.** *For $m = 2k$ we have*

$$\gamma_{2k}(\omega)\, s^{(2k)} \equiv h^{(2k)} \mod \mathrm{J}_{2k}.$$

*Proof.* We will start with the formula (2.3) for $s^{(2k)}$ and use an inductive procedure to apply a sequence of reductions modulo $\mathrm{J}_{2k}$ to eliminate all diagrams containing horizontal edges from the sum. As a first step, for each $r = 0,1,\dots,k$ split the set of diagrams $\mathcal{D}^{(r)}$ into three subsets,

$$\mathcal{D}^{(r)} = \mathcal{D}^{(r,-)} \cup \mathcal{D}^{(r,0)} \cup \mathcal{D}^{(r,+)}, \tag{2.15}$$

where $d \in \mathcal{D}^{(r,-)}$ if and only if the vertices $1$ in the top and bottom rows are the ends of horizontal edges; $d \in \mathcal{D}^{(r,+)}$ if and only if the vertices $1$ are the ends of different non-horizontal edges, and the remaining diagrams belong to $\mathcal{D}^{(r,0)}$. In particular, $\mathcal{D}^{(0)} = \mathcal{D}^{(0,+)}$ and $\mathcal{D}^{(k)} = \mathcal{D}^{(k,-)}$. It is clear by the application of the transposition $\mathrm{t}_1$ that for $r \geqslant 0$

$$\sum_{d\in\mathcal{D}^{(r,0)}} d \equiv 0 \mod \mathrm{J}_{2k} \qquad \text{and} \qquad \sum_{d\in\mathcal{D}^{(r+1,-)}} d \equiv -\sum_{d\in\mathcal{D}^{(r,+)}} d \mod \mathrm{J}_{2k}.$$

Taking into account the relation

$$\binom{\omega/2+2k-2}{r}^{-1} + \binom{\omega/2+2k-2}{r+1}^{-1} = \frac{\omega+4k-2}{\omega+4k-4}\binom{\omega/2+2k-3}{r}^{-1},$$

we can conclude from (2.3) that the reduction modulo $\mathrm{J}_{2k}$ yields the equivalence

$$\gamma_{2k}(\omega)\, s^{(2k)} \equiv \frac{\gamma_{2k-2}(\omega+2)}{(2k)!} \sum_{r=0}^{k-1} (-1)^r \binom{\omega/2+2k-3}{r}^{-1} \sum_{d \in \mathcal{D}^{(r,+)}} d. \tag{2.16}$$

Note that the inverse binomial coefficients in this expression coincide with those in (2.3) for $m = 2k-2$ with the parameter $\omega$ replaced with $\omega + 2$.

For the second step of the reduction, represent each set $\mathcal{D}^{(r,+)}$ as the union

$$\mathcal{D}^{(r,+)} = \bigcup_{a,b=2}^{k} \mathcal{D}^{(r,+)}_{a,b},$$

where the subset $\mathcal{D}^{(r,+)}_{a,b}$ consists of the diagrams $d$ containing the (non-horizontal) edges $(1,a)$ and $(1,b)$. Re-arrange expression (2.16) to include the extra internal sum over the diagrams $d \in \mathcal{D}^{(r,+)}_{a,b}$ with fixed values of $a$ and $b$. If $a = b$, then by ignoring the vertices $1$ and $a$ in the top and bottom rows, we get the desired part of the expression modulo $\mathrm{J}_{2k}$ by the induction hypothesis. If $a \neq b$, then split the union $\mathcal{D}^{(r,+)}_{a,b} \cup \mathcal{D}^{(r,+)}_{b,a}$ as in (2.15),

$$\mathcal{D}^{(r,+)}_{a,b} \cup \mathcal{D}^{(r,+)}_{b,a} = \mathcal{D}^{(r,+,-)}_{\{a,b\}} \cup \mathcal{D}^{(r,+,0)}_{\{a,b\}} \cup \mathcal{D}^{(r,+,+)}_{\{a,b\}},$$

where $d \in \mathcal{D}^{(r,+,-)}_{\{a,b\}}$ if and only if the remaining vertices $a$ and $b$ are the ends of horizontal edges; $d \in \mathcal{D}^{(r,+,+)}_{\{a,b\}}$ if and only if the remaining vertices $a$ and $b$ are the ends of different non-horizontal edges, and the remaining diagrams belong to $\mathcal{D}^{(r,+,0)}_{\{a,b\}}$. Similar to the first reduction step, the application of the composition of transpositions $\mathrm{t}_1 \circ \mathrm{t}_a \circ \mathrm{t}_b$ shows that for $r \geqslant 0$

$$\sum_{d \in \mathcal{D}^{(r,+,0)}_{\{a,b\}}} d \equiv 0 \mod \mathrm{J}_{2k} \qquad \text{and} \qquad \sum_{d \in \mathcal{D}^{(r+1,+,-)}_{\{a,b\}}} d \equiv -\sum_{d \in \mathcal{D}^{(r,+,+)}_{\{a,b\}}} d \mod \mathrm{J}_{2k}.$$

This leads to the second step reduction formula analogous to (2.16), and the argument continues in the same way by an obvious induction. □

The required relation (2.14) follows from Lemma 2.3 because the transposition $\mathrm{t}_a$ on the Brauer algebra is consistent with the partial transposition $\mathrm{t}_a$ on the tensor product (2.2). That is, if an element $s \in \mathcal{B}_m(\omega)$ has the form $s = d + d^{\,\mathrm{t}_a}$, then for the image $S$ of $s$ under the respective actions (2.6) and (2.8) we have

$$\mathrm{tr}_{1,\dots,m}\, S\, F[-p_1]_1 \dots F[-p_m]_m = 0,$$

since $S$ is stable the transposition $\mathrm{t}_a$, while $F[-p_a] + F[-p_a]^{\mathrm{t}} = 0$. □

*Remark* 2.4. (i) The equality of the top degree components in (2.14) is a relation for elements of the symmetric algebra $\mathrm{S}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ and it follows by comparing the Chevalley images of both sides; see [8, Prop. 2.2.4 & Cor. 2.2.10].

(ii) By [8, Lemma 1.3.4], for the partial trace of the element $h^{(2k)} \in \mathcal{B}_{2k}(\omega)$ we have

$$\operatorname{tr}_{2l+1,\dots,2k} h^{(2k)} = \binom{\omega+2k-1}{2k-2l}\binom{2k}{2l}^{-1} h^{(2l)}.$$

Since the coefficient of $h^{(2l)}$ differs from that in formula (2.13) only by the shift $\omega \mapsto \omega - 1$, one can write an equivalent formula in the symbolic form

$$\phi_{2k} = \operatorname{tr}_{1,\dots,2k} \overline{H}^{(2k)} \mathcal{F}_1 \dots \mathcal{F}_{2k}, \tag{2.17}$$

where the symbol $\overline{H}^{(m)}$ is interpreted via an expansion similar to that of (2.9), with the partial traces calculated by the rule

$$\operatorname{tr}_m \overline{H}^{(m)} = \frac{\omega+m-2}{m} \overline{H}^{(m-1)}$$

for $\omega = N$ or $-N$, respectively. In the symplectic case, this can be interpreted more formally by calculating the right hand side of (2.17) on the space (2.2) with $N = 2n+1$, where $\overline{H}^{(m)}$ is understood as the anti-symmetrization operator, while $F[-1]$ is regarded as the $N \times N$ matrix whose entries in the last row and column are zero.

(iii) For any nonzero $z \in \mathbb{C}$, the image of a complete set of Segal–Sugawara vectors under the evaluation homomorphism

$$\mathrm{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big) \to \mathrm{U}(\mathfrak{g}), \qquad X[r] \mapsto X\, z^r,$$

with $X \in \mathfrak{g}$ and $r < 0$, is a set of algebraically independent generators of the center of $\mathrm{U}(\mathfrak{g})$; see [8, Prop. 6.5.2]. Therefore, Theorem 2.2 provides new formulas for such generators in types $B$, $C$ and $D$ by applying the evaluation homomorphism to the Segal–Sugawara vectors $\phi_{2k}$. $\square$

Finally, we will make a connection with the results of [11] in the symplectic case and so will now take $\mathfrak{g} = \mathfrak{sp}_{2n}$. The elements $\varpi(\Delta_{2k}[-1]) \in \mathrm{U}\big(t^{-1}\mathfrak{sp}_{2n}[t^{-1}]\big)$ were introduced therein via the symmetrization map $\varpi$, and in our notation they can be written as

$$\varpi(\Delta_{2k}[-1]) = \operatorname{tr}_{1,\dots,2k} H^{(2k)} F[-1]_1 \dots F[-1]_{2k},$$

for $k = 1, \dots, n$. As above, expand the product $\mathcal{F}_1 \dots \mathcal{F}_m$ as a linear combination of monomials of the form (2.10). By applying conjugations by suitable permutations of the index set $1, \dots, m$ and using the cyclic property of trace, we get the expansion

$$\operatorname{tr}_{1,\dots,m} H^{(m)} \mathcal{F}_1 \dots \mathcal{F}_m = \sum_{s=1}^{m} \operatorname{tr}_{1,\dots,m} H^{(m)} \Psi_s^{(m)},$$

so that $\Psi_s^{(m)}$ is a certain linear combination of monomials of the form $F[-r_1]_1 \dots F[-r_s]_s$, where $r_1 + \dots + r_s = m$ and $r_i \geqslant 1$. By evaluating partial traces, we get

$$\operatorname{tr}_{1,\dots,2l} H^{(2l)} \Psi_{2l}^{(2k)} = \binom{2k}{2l} \varpi\big(T^{2k-2l} \Delta_{2l}[-1]\big).$$

The proof of Theorem 2.2 shows that formula (2.13) will remain valid if $\mathrm{tr}_{1,\dots,2l}\, H^{(2l)}\Phi^{(2k)}_{2l}$ is replaced with $\mathrm{tr}_{1,\dots,2l}\, H^{(2l)}\Psi^{(2k)}_{2l}$. Therefore, taking $\omega = -2n$ in the formula, we derive the identity

$$\phi_{2k} = \sum_{l=1}^{k} \binom{2n-2l+1}{2k-2l} \varpi\Big(T^{2k-2l}\Delta_{2l}[-1]\Big),$$

so that $\phi_{2k}$ coincides with the Segal–Sugawara vector given by [11, Theorem 4.4].

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