

NILPOTENT GROUPS WITH BALANCED PRESENTATIONS. II

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ABSTRACT. Let G be a nilpotent group with a balanced presentation. Then wither $G \cong \mathbb{Z}^3$ or $\beta_1(G; \mathbb{Q}) \leq 2$. We show that if G has an abelian normal subgroup A such that $G/A \cong \mathbb{Z}^2$ then G is torsion-free and has Hirsch length $h(G) \leq 4$. We also consider the torsion subgroup of G when $h(G) \leq 2$.

A finite presentation for a group G is *balanced* if it has an equal number of generators and relations. This notion has been mostly studied in connection with finite groups, but here we shall consider finitely generated infinite nilpotent groups. In [8] we showed that if such a group G has a balanced presentation then either $G \cong \mathbb{Z}^3$ or $\beta_1(G; \mathbb{Q}) \leq 2$, and if G is also metabelian then it has Hirsch length $h(G) \leq 4$. There are nilpotent groups with balanced presentations and which have non-trivial torsion, but all known examples have Hirsch length $h \leq 1$. We expect that nilpotent groups with balanced presentations and Hirsch length $h > 1$ should be torsion-free. Our methods confirm this for G metabelian, nilpotent and with torsion-free abelianization.

We shall describe briefly the topological context from which this work arose [5, 6, 7]. The complement of an embedding of a closed 3-manifold M in the 4-sphere S^4 has two components, with closures X and Y , say, and $\chi(X) + \chi(Y) = 2$. We may assume that $\chi(X) \leq \chi(Y)$, and so $\chi(X) \leq 1$. The inclusions of M into X and Y induce isomorphisms $H_i(M; \mathbb{Z}) \cong H_i(X; \mathbb{Z}) \oplus H_i(Y; \mathbb{Z})$ for $i = 1, 2$. If moreover $\pi_X = \pi_1(X)$ and $\pi_Y = \pi_1(Y)$ are each nilpotent then the maps on fundamental groups are epimorphisms, and we then say that the embedding is *nilpotent*. There are strong constraints on nilpotent embeddings. Either $\pi_X \cong \mathbb{Z}$ and $\pi_Y = 1$ or $\pi_X \cong \mathbb{Z}^2$ and $\pi_Y \cong \mathbb{Z}$, or $\chi(X) = \chi(Y) = 1$. (See Theorem 13 below). In the latter case $\beta_2(\pi_X; F) \leq \beta_1(\pi_X; F)$ for all field coefficients F . Hence either $\pi_X \cong \pi_Y \cong \mathbb{Z}^3$ or $\beta_1(\pi_X; \mathbb{Q}) = \beta_1(\pi_Y; \mathbb{Q}) \leq 2$ [7, 8]. Whether there are infinitely many such groups with Hirsch length $h(\pi_X) > 3$ is unknown. On the constructive side, if groups G and H have balanced presentations and isomorphic abelianizations then there is an embedding with $\pi_X \cong G$ and $\pi_Y \cong H$ [10].

If a group G has a balanced presentation then $\beta_2(G; F) \leq \beta_1(G; F)$ for all fields F . Although this is not a sufficient condition for G to have a balanced presentation, it is the most useful necessary condition and underlies all of our arguments. In [8] the emphasis was on the rational field $F = \mathbb{Q}$, and the torsion subgroup could be ignored. Here we are interested in delineating the possible torsion, and so other coefficient fields are needed. Our strategy shall be to show that if a finitely generated nilpotent group G has a normal subgroup K with quotient \mathbb{Z}^r , for $r = 1$ or 2, and $\beta_2(G; \mathbb{F}_p) \leq \beta_1(G; \mathbb{F}_p)$ then $H^2(K; \mathbb{F}_p)$ has no subgroup which is a direct sum of non-trivial subgroups invariant under the action of $Aut(K)$. This test seems

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difficult to apply, except when K is abelian, and so our main results are restricted to groups which are abelian extensions of \mathbb{Z}^r .

The first section presents our notation and some basic facts, while in §2 we summarize briefly what is currently known about finite nilpotent groups with balanced presentations. We have little to add to the known results [2, 3]. The next three sections prepare for working with infinite nilpotent groups. In §6 we show that if G has a balanced presentation and $h(G) = 1$ then G is 2-generated, and if the torsion subgroup of G is abelian then $G \cong \mathbb{Z}/m\mathbb{Z} \rtimes_n \mathbb{Z}$, for some $m, n \geq 1$ (Theorem 15). In §7 we show that if G has an abelian normal subgroup A such that $G/A \cong \mathbb{Z}^2$ then G is torsion-free and A has rank ≤ 2 (Corollary 25). While most of our results are framed in purely algebraic terms, we shall illustrate them by reference to embeddings in S^4 .

1. NOTATION AND GENERALITIES

If G is a group then G' , $G^{ab} = G/G'$, ζG and $|G|$ shall denote the commutator subgroup, abelianization, centre and order of G , respectively. The Hirsch length $h(S)$ of a solvable group S is the sum of the ranks of the abelian sections of a composition series for S . If N is a finitely generated nilpotent group then it is finitely presentable and $h(N)$ is finite.

A group is *d-generated* if it can be generated by d elements and has a *balanced presentation* if it has a finite presentation with equal numbers of generators and relations. We shall say that G is *homologically balanced* if G is finitely generated and $\beta_2(G; F) \leq \beta_1(G; F)$, for all fields F . If G has a balanced presentation then it is homologically balanced, and if G is also finite then it must have trivial multiplier: $H_2(G; \mathbb{Z}) = 0$. (These assertions follow most easily from consideration of the homology of the 2-complex associated to a balanced presentation for the group.)

If G is finitely presentable then $H_i(G; R)$ is finitely generated for $i \leq 2$ and all simple coefficients R . It follows easily that $\beta_i(G; \mathbb{Q}) = \beta_i(G; \mathbb{F}_p)$, for $i \leq 2$ and almost all primes p . Hence $\beta_2(G; \mathbb{Q}) = \beta_1(G; \mathbb{Q})$ if and only if $\beta_2(G; \mathbb{F}_p) = \beta_1(G; \mathbb{F}_p)$, for almost all primes p .

The Universal Coefficient Theorem for homology gives an exact sequence

$$0 \rightarrow F \otimes H_2(G; \mathbb{Z}) \rightarrow H_2(G; F) \rightarrow \text{Tor}(F, G^{ab}) \rightarrow 0,$$

for any group G and field F , since $H_1(G; \mathbb{Z}) = G^{ab}$. If A is a finitely generated abelian group and $F = \mathbb{F}_p$ then $\text{Tor}(\mathbb{F}_p, A) \cong {}_pA = \text{Ker}(p.id_A)$. If A is finite then A/pA and $\text{Ker}(p.id_A)$ have the same dimension. Hence if G is finite then $\beta_2(G; \mathbb{F}_p) \geq \beta_1(G; \mathbb{F}_p)$, for any prime p , and G is homologically balanced if and only if $H_2(G; \mathbb{Z}) = 0$.

If G is abelian then $G = A$ and $H_2(G; \mathbb{Z}) = A \wedge A$. If also $F = \mathbb{F}_p$, and p is odd or if $p = 2$ and A has no summand of exponent 2 then this sequence is canonically split, and so $H_2(A; \mathbb{F}_p) \cong ((A/pA) \wedge (A/pA)) \oplus \text{Ker}(p.id_A)$ [1, Theorem V.6.6]. There are abelian 2-groups for which there is no canonical splitting [9].

There are similar Universal Coefficient exact sequences for cohomology

$$0 \rightarrow \text{Ext}(G^{ab}, F) \rightarrow H^2(G; F) \rightarrow \text{Hom}(G^{ab}, F) \rightarrow 0.$$

We shall assume throughout that $\pi_X = \pi_1(X)$ and $\pi_Y = \pi_1(Y)$, where X and Y are the closures of the components of the complement of a hypersurface in S^4 , and that $\chi(X) \leq 1$. Other groups shall be denoted by Greek or Roman capitals.

2. $h = 0$: FINITE GROUPS

A nilpotent group G is finite if and only if $\beta_1(G; \mathbb{Q}) = 0$ if and only if $h(G) = 0$. The Sylow subgroups of a finite nilpotent group G are characteristic, and G is the direct product of its Sylow subgroups [12, 5.2.4]. It then follows from the Künneth Theorem that $H_2(G; \mathbb{Z}) = 0$ if and only if $H_2(P; \mathbb{Z}) = 0$ for all such Sylow subgroups P . On the other hand, it is not clear that if $H_2(G; \mathbb{Z}) = 0$ then N must have a balanced presentation, even if this is so for each of its Sylow subgroups. In general there may be a gap between homological necessary conditions and combinatorial sufficient conditions. (The examples in [2] of finite groups with trivial multiplier but without balanced presentations are not nilpotent.)

Lemma 1. [11] *If G is a finitely generated nilpotent group and $\beta_2(G; F) \leq \beta_1(G; F)$ for all field coefficients F then G is 3-generated.* \square

If π_X is nilpotent then $c.d.X \leq 2$ [5, Theorem 5.1]. Therefore the singular chain complex of the universal cover \tilde{X} is chain homotopy equivalent to a finite free $\mathbb{Z}[\pi_X]$ -complex of length 2. Hence the augmentation ideal of the group ring $\mathbb{Z}[\pi_X]$ has a square presentation matrix, since $\chi(X) \leq 1$. This property interpolates between π_X having a balanced presentation and having $\beta_2(\pi_X; F) \leq \beta_1(\pi_X; F)$ for all field coefficients F . The stronger condition (having a balanced presentation) would hold if X were homotopy equivalent to a finite 2-dimensional cell complex.

Theorem 2. *If π_X is nilpotent and $H_1(Y; \mathbb{Z})$ is a non-trivial finite group then $H_2(X; \mathbb{Z}) = H_2(Y; \mathbb{Z}) = 0$. Hence $\chi(X) = 1$, π_X is finite and $H_2(\pi_X; \mathbb{Z}) = H_2(\pi_Y; \mathbb{Z}) = 0$.*

Proof. Since π_X has no noncyclic free subgroup, $\chi(X) \geq 0$, and $c.d.\pi_X \leq 2$ if $\chi(X) = 0$ [6, Theorem 2]. Thus if $\chi(X) = 0$ then $\pi_X = 1, \mathbb{Z}$ or \mathbb{Z}^2 , and $H_1(X; \mathbb{Z})$ is torsion-free. But the torsion subgroups of $H_1(X; \mathbb{Z})$ and $H_1(Y; \mathbb{Z})$ are isomorphic. Hence $\chi(X) > 0$, and so $\chi(X) = \chi(Y) = 1$, since $\chi(X) \leq \chi(Y) = 2 - \chi(X)$. Therefore $H^1(X; \mathbb{Z}) \cong H_2(Y; \mathbb{Z}) = 0$ and $H_2(X; \mathbb{Z}) \cong H^1(Y; \mathbb{Z}) = 0$. Hence $\pi_X^{ab} = H_1(X; \mathbb{Z})$ is finite. Since π_X is nilpotent and has finite abelianization, it is finite. Moreover, $H_2(\pi_X; \mathbb{Z}) = H_2(\pi_Y; \mathbb{Z}) = 0$, since these groups are quotients of $H_2(X; \mathbb{Z})$ and $H_2(Y; \mathbb{Z})$, respectively. \square

The assumption in the theorem that $H_1(Y; \mathbb{Z})$ be non-trivial is essential. The closures of the complementary regions of the standard embedding of $S^2 \times S^1$ in S^4 are $X = D^3 \times S^1$ and $Y = S^2 \times D^2$, with fundamental groups \mathbb{Z} and 1, respectively.

A finite abelian group A has a balanced presentation if and only if it is cyclic, for if A/pA is not cyclic for some prime p then $(A/pA) \wedge (A/pA) \neq 0$, and so $\beta_2(A; \mathbb{F}_p) > \beta_1(A; \mathbb{F}_p)$, by the Universal Coefficient exact sequences of §1.

If p is an odd prime then every 2-generator metacyclic p -group P with $H_2(P; \mathbb{Z}) = 0$ has a balanced presentation

$$\langle a, b \mid b^{p^{r+s+t}} = a^{p^{r+s}}, bab^{-1} = a^{1+p^r} \rangle,$$

where $r \geq 1$ and $s, t \geq 0$. (The order of such a group is $p^{3r+2s+t}$.) There are other metacyclic 2-groups and other p -groups with 2-generator balanced presentations. A handful of 3-generated p -groups (for $p = 2$ and 3) are also known to have balanced presentations. (See [3] for a survey of what was known in the mid-1990s.)

The finite nilpotent 3-manifold groups $Q(8k) \times \mathbb{Z}/a\mathbb{Z}$ (with $(a, 2k) = 1$) have the balanced presentations

$$\langle x, y \mid x^{2ka} = y^2, yxy^{-1} = x^s \rangle,$$

where $s \equiv 1 \pmod{a}$ and $s \equiv -1 \pmod{2k}$. The other finite nilpotent groups F with 4-periodic cohomology (the generalized quaternionic groups $Q(2^n a, b, c) \times \mathbb{Z}/d\mathbb{Z}$, with a, b, c, d odd and pairwise relatively prime) have $H_2(F; \mathbb{Z}) = 0$, but we do not know whether they all have balanced presentations.

3. UNIPOTENT AUTOMORPHISMS

An automorphism α of an abelian group A is *unipotent* if $\alpha - id_A$ is nilpotent.

Lemma 3. *Let N be a finitely generated nilpotent group and ψ an automorphism of N . Then $G = N \rtimes_{\psi} \mathbb{Z}$ is nilpotent if and only if ψ^{ab} is unipotent.*

Proof. The necessity of the condition is clear. Suppose that it holds. If N is abelian then it is clear that G is nilpotent. We shall induct on the length of a composition series for N . The terms $\gamma_i N$ of the lower central series of N are characteristic, and so are invariant under ψ . It is easily seen that if α is a unipotent automorphism of an abelian group A then $\otimes^i \alpha$ is a unipotent automorphism of $\otimes^i A$ for all i . Hence it follows from [12, 5.2.5] that the induced automorphisms of the subquotients $\gamma_i N / \gamma_{i+1} N$ are unipotent. Since the last such term is central in N it is abelian, and so ψ fixes a nontrivial subgroup $C \leq \zeta N$. Hence ζG is nontrivial. Since $N/\zeta G$ has a shorter composition series than N , it follows from the inductive hypothesis that $G/\zeta G$ is nilpotent. Hence G is nilpotent. \square

We shall extend the term “unipotent”, to say that an automorphism ψ of a finitely generated nilpotent group is unipotent if ψ^{ab} is unipotent. Furthermore, an action $\alpha : G \rightarrow \text{Aut}(A)$ is unipotent if $\alpha(g)$ is unipotent for all $g \in G$.

Lemma 4. *Let N be a finitely generated nilpotent group which acts unipotently on a finitely generated abelian group A , and let \mathfrak{n} be the augmentation ideal of $\mathbb{Z}[N]$. Then A has a finite filtration $A = A_1 > \dots > A_k = A^N > A_{k+1} = 0$ by $\mathbb{Z}[N]$ -submodules, where A^N is the fixed subgroup and $\mathfrak{n}A_i \leq A_{i+1}$, for $i \leq k$.*

Proof. We induct on the length of the upper central series of N . The centre ζN is a nontrivial abelian group which acts unipotently on A , and it is easy to see that $A^{\zeta N} \neq 0$. The quotient $N/\zeta N$ acts unipotently on each of $A^{\zeta N}$ and $\bar{A} = A/A^{\zeta N}$, and so these each have such filtrations, by the inductive hypothesis. The preimages of the filtration of \bar{A} in A combine with the filtration of $A^{\zeta N}$ to give the required filtration. \square

It is easy to see that the product of commuting unipotent automorphisms is unipotent. This observation extends to show that an action of a nilpotent group N is unipotent if N is generated by elements which act unipotently.

Our next lemma is probably known, but we have not found a published proof.

Lemma 5. *Let ψ be a unipotent automorphism of a finitely generated nilpotent group N . Then $H_i(\psi; R)$ and $H^i(\psi; R)$ are unipotent, for all simple coefficients R and $i \geq 0$.*

Proof. If N is cyclic then the result is clear. In general, N has a composition series with cyclic subquotients $\mathbb{Z}/p\mathbb{Z}$, where $p = 0$ or is prime. We shall induct on the number of terms in such a composition series. If N is infinite then ψ acts unipotently on $\text{Hom}(N, \mathbb{Z})$ and so fixes an epimorphism to \mathbb{Z} ; if N is finite then ψ fixes an epimorphism to $\mathbb{Z}/p\mathbb{Z}$, for any p dividing the order of N .

Let K be the kernel of such an epimorphism. Then $\psi(K) = K$, by the choice of ψ ; let $\psi_K = \psi|_K$. This is a unipotent automorphism of K , by the criterion of Lemma 3. Hence the induced action of ψ on $H_i(K; R)$ is unipotent, for all i , by the inductive hypothesis. Let $\Lambda = \mathbb{Z}[N/K]$ and let B be a Λ -module. Then $H_i(N/K; B) = \text{Tor}_i^\Lambda(\mathbb{Z}, B)$ may be computed from the tensor product $C_* \otimes_{\mathbb{Z}} B$, where C_* is a resolution of the augmentation Λ -module \mathbb{Z} . If $B = H_i(K; R)$ then the diagonal action of ψ on each term of $C_* \otimes_{\mathbb{Z}} B$ is unipotent. The result is now a straightforward consequence of the Lyndon-Hochschild-Serre spectral sequence for N as an extension of N/K by K .

The argument for cohomology is similar. \square

In fact we only need this lemma in degrees ≤ 2 . We shall usually assume that the coefficient ring is a field, and then homology and cohomology are linear duals of each other. Homology has an advantage deriving from the isomorphism $G^{ab} \cong H_1(G; \mathbb{Z})$, but it is often more convenient to use cohomology instead.

4. UNIPOTENT ACTIONS ON ABELIAN GROUPS

We shall find the following notion useful in many of our arguments. Let G be a group and F a field. Then an $F[G]$ -module V is *canonically subsplit* if it contains a nontrivial direct sum of $F[G]$ -submodules. If G acts unipotently on V and V is canonically subsplit then the subspaces of the summands fixed by G/K are non-trivial, by Lemma 4, and so the subspace V^G fixed by G has dimension > 1 .

Lemma 6. *Let A be a finitely generated abelian group and p a prime such that A has non-trivial p -torsion and $\dim_{\mathbb{F}_p} A/pA > 1$. If p is odd or if $p = 2$ and A has no $\mathbb{Z}/2\mathbb{Z}$ summand then $H_2(A; \mathbb{F}_p)$ and $H^2(A; \mathbb{F}_p)$ are each canonically subsplit with respect to the natural action of (subgroups of) $\text{Aut}(A)$.*

Proof. Let $W = (A/pA) \wedge (A/pA)$ and $A^* = \text{Hom}(A; \mathbb{F}_p) = H^1(A; \mathbb{F}_p)$.

Then there is a natural splitting $H_2(A; \mathbb{F}_p) = W \oplus \text{Tor}(A, \mathbb{F}_p)$ if p is odd [1, Chapter V.6], or if $p = 2$ and A has no $\mathbb{Z}/2\mathbb{Z}$ summand [9]. There is also a natural epimorphism $\theta : H^2(A; \mathbb{F}_p) \rightarrow \text{Hom}(W, \mathbb{F}_p)$, with kernel isomorphic to $\text{Ext}(A; \mathbb{F}_p)$ [1, Exercises IV.3.8 and V.6.5].

If p is odd then cup product induces a monomorphism $c_A : A^* \wedge A^* \rightarrow H^2(A; \mathbb{F}_p)$, since A is abelian. If $p = 2$ then cup product defines a homomorphism from $A^* \odot A^*$ to $H^2(A; \mathbb{F}_2)$. Since A has no $\mathbb{Z}/2\mathbb{Z}$ summand, $Sq(a) = a \cup a = 0$ for all $a \in A^*$, and so cup product again induces a monomorphism $c_A : A^* \wedge A^* \rightarrow H^2(A; \mathbb{F}_p)$ [4]. It is easily seen from the formulae in [1] that $\theta \circ c_A$ is an isomorphism, and so $H^2(A; \mathbb{F}_p)$ is naturally isomorphic to $(A^* \wedge A^*) \oplus \text{Ext}(A; \mathbb{F}_p)$.

The summands are all non-trivial, since A has nontrivial p -torsion and A/pA is not cyclic. Thus $H_2(A; \mathbb{F}_p)$ and $H^2(A; \mathbb{F}_p)$ are each canonically subsplit. \square

The case when A has a summand of exponent 2 seems more complicated, and we consider only the cohomology.

Lemma 7. *Let A be a finitely generated abelian group with a nontrivial summand of exponent 2 and such that $\dim_{\mathbb{F}_2} A/2A > 1$. Suppose that a finitely generated nilpotent group N acts unipotently on A . Then $\dim_{\mathbb{F}_2} H^2(A; \mathbb{F}_2)^N > 1$.*

Proof. We may assume that $A \cong B \oplus E$, where $E \cong (\mathbb{Z}/2\mathbb{Z})^s \neq 0$ and B has no summand of order 2. The subspace B^* of $A^* = \text{Hom}(A, \mathbb{F}_2) = H^1(A; \mathbb{F}_2)$ consisting of homomorphisms which factor through homomorphisms to $\mathbb{Z}/4\mathbb{Z}$ is canonical. Clearly $B^* \cong \text{Hom}(B, \mathbb{F}_2)$ and $A^*/B^* \cong E^* = \text{Hom}(E, \mathbb{F}_2)$. Hence $A^* \cong B^* \oplus E^*$, but this splitting is not canonical. Cup product induces a homomorphism $c_A : A^* \odot A^* \rightarrow H^2(A; \mathbb{F}_2)$, with kernel $2A/4A \cong B^*$, since A is abelian [4]. There is also a natural squaring map $Sq : A^* \rightarrow H^2(A; \mathbb{F}_2)$ with kernel B^* .

If $B = 0$ then A is an elementary 2-group and $A^* = E^*$, and c_A is a monomorphism. Let $A_1 > \cdots > A_{k+1} = 0$ be a filtration of A^* by $\mathbb{F}_p[N]$ -submodules, as in Lemma 4. Then $A_k \odot A_k$ is fixed by N . If $\dim_{\mathbb{F}_2} A_k > 1$ then $\dim_{\mathbb{F}_2} A_k \odot A_k \geq 3$. If A_k has dimension 1, and is generated by b then $b \odot b$ is fixed by N . If $a \in A_{k-1}$ then each element of N either fixes a or sends it to $a+b$. In either case $a \odot (a+b)$ is fixed by N . Since $\dim_{\mathbb{F}_2} A_{k-1} \geq 2$ the subspace generated by $\{a \odot (a+b) \mid a \in A_{k-1}\} \cup \{b \odot b\}$ is fixed by N , and so $\dim_{\mathbb{F}_2} H^2(A; \mathbb{F}_2)^N > 1$.

The images of $B^* \odot A^*$ and $Sq(A^*) = Sq(E^*)$ are canonical submodules of $H^2(A; \mathbb{F}_2)$, with trivial intersection. Hence they are invariant under the action of automorphisms of A , and so if $B \neq 0$ then we again have $\dim_{\mathbb{F}_2} H^2(A; \mathbb{F}_2)^N > 1$. \square

Corollary 8. *Let A be a finitely generated abelian group, ψ be a unipotent automorphism of A , and p be a prime. If A has non-trivial p -torsion and $\dim_{\mathbb{F}_p} A/pA > 1$ then $\dim_{\mathbb{F}_p} \text{Ker}(H_2(\psi; \mathbb{F}_p) - I) = \dim_{\mathbb{F}_p} \text{Ker}(H^2(\psi; \mathbb{F}_p) - I) > 1$.*

Proof. Let N be the cyclic subgroup of $\text{Aut}(A)$ generated by ψ . We shall write $H_i(\psi)$ and $H^j(\psi)$ instead of $H_i(\psi; \mathbb{F}_p)$ and $H^j(\psi; \mathbb{F}_p)$, for simplicity of notation. Then $H_i(A; \mathbb{F}_p)^N = \text{Ker}(H_i(\psi) - I)$ and $H^j(A; \mathbb{F}_p)^N = \text{Ker}(H^j(\psi) - I)$, for any i . If φ is an endomorphism of a finite dimensional vector space V then $\dim \text{Cok}(\varphi) = \dim \text{Ker}(\varphi)$ and if φ^* is the induced endomorphism of the dual vector space V^* then φ^* and φ have the same rank. Hence the corollary follows from Lemma 6, if p is odd, and from Lemma 7, if $p = 2$. \square

It does not seem obvious that $\dim_{\mathbb{F}_p} H_2(A; \mathbb{F}_p)^N$ and $\dim_{\mathbb{F}_p} H^2(A; \mathbb{F}_p)^N$ are equal when N is not cyclic.

If $\dim_{\mathbb{F}_p} A/pA \geq 4$ then the restriction of $H_2(\psi; \mathbb{F}_p) - I$ to $(A/pA) \wedge (A/pA)$ has kernel of dimension > 1 , and so $\dim_{\mathbb{F}_p} \text{Ker}(H_2(\psi; \mathbb{F}_p) - I) > 1$. In [8] a related observation for free abelian groups of rank ≥ 4 is used to show that if G is a metabelian nilpotent group with $h(G) > 4$ then $\beta_2(G; \mathbb{Q}) > \beta_1(G; \mathbb{Q})$.

One of the difficulties in extending the approach of this paper to more general nilpotent groups is the lack of an analogue to the above lemmas for non-abelian p -groups. If T is one of the 2-generator metacyclic p -groups of §2 then $H^2(T; \mathbb{F}_p)$ has no canonically split subspace, and such groups do arise as the torsion subgroups of homologically balanced nilpotent groups G with $h(G) = 1$. (See the final paragraphs of §5 below.) Can we at least use such an argument to show that the torsion subgroup must be homologically balanced?

5. WANG SEQUENCE ESTIMATES

If G is a finitely generated infinite nilpotent group then there is an epimorphism $f : G \rightarrow \mathbb{Z}$, and so $G \cong K \rtimes_{\psi} \mathbb{Z}$, where ψ is an automorphism of $K = \text{Ker}(f)$ determined by conjugation in G . The homology groups $H_i(K; R) = H_i(G; R[G/K])$ are $R[G/K]$ -modules, with a generator t of $G/K \cong \mathbb{Z}$ acting via $H_i(\psi; R)$. The long exact sequence of homology associated to the short exact sequence of coefficients

$$0 \rightarrow R[G/K] \xrightarrow{t-1} R[G/K] \rightarrow R \rightarrow 0$$

is the *Wang sequence*

$$\begin{aligned} H_2(K; R) \xrightarrow{H_2(\psi; R) - I} H_2(K; R) \rightarrow H_2(G; R) \rightarrow H_1(K; R) \xrightarrow{H_1(\psi; R) - I} H_1(K; R) \rightarrow \\ \rightarrow H_1(G; R) \rightarrow R \rightarrow 0. \end{aligned}$$

There is a similar Wang sequence for cohomology. (These are special cases of the Lyndon-Hochschild-Serre spectral sequences for the homology and cohomology with coefficients R of G as an extension of \mathbb{Z} by K .)

Lemma 9. *Let $G \cong K \rtimes_{\psi} \mathbb{Z}$ be a finitely generated nilpotent group, and let F be a field. Then*

- (1) $\dim_F \text{Cok}(H_2(\psi; F) - I) = \dim_F \text{Ker}(H^2(\psi; F) - I) = \beta_2(G) - \beta_1(G) + 1$,
and so $\beta_2(G; F) \geq \beta_1(G; F) - 1$, with equality if and only if $\beta_2(K; F) = 0$;
- (2) if $\beta_2(G; F) = \beta_1(G; F)$ then $H_2(K; F)$ is cyclic as a $F[G/K]$ -module;
- (3) if $\beta_1(G; F) = 1$ then $\beta_1(K; F) = 0$, K is finite, $h(G) = 1$ and $\beta_2(G; F) = 0$;
- (4) if $H_2(G; \mathbb{Z}) = 0$ then $G \cong \mathbb{Z}$.

Proof. Part (1) follows from the Wang sequences for the homology and cohomology of G as an extension of \mathbb{Z} by K . The endomorphisms $H_i(\psi; F) - I$ have non-trivial kernel and cokernel if $H_i(K; F) \neq 0$, since they are nilpotent, by Lemma 5.

The $F[G/K]$ -module $H = H_2(K; F)$ is finitely generated and is annihilated by a power of $t - 1$, since $H_2(\psi; F)$ is unipotent. If $\beta_2(G; F) = \beta_1(G; F)$ then $\dim_F H/(t - 1)H = 1$, by the exactness of the Wang sequence. Since $F[G/K] \cong F[t, t^{-1}]$ is a PID, it follows that H is cyclic as an $F[G/K]$ -module.

Let $t \in G$ represent a generator of G/K . Then $F[G/K] \cong F[t, t^{-1}]$ is a PID and $H = H_2(K; F) = H_2(G; F[G/K])$ is a finitely generated $F[t, t^{-1}]$ -module, with t acting via $H_2(\psi; F)$. This module is annihilated by a power of $t - 1$, since $H_2(\psi; F)$ is unipotent. If $\beta_2(G; F) = \beta_1(G; F)$ then $\dim_F H/(t - 1)H = 1$, by exactness of the Wang sequence. Since $F[G/K] \cong F[t, t^{-1}]$ is a PID, it follows that H is cyclic as an $F[G/K]$ -module.

If $\beta_1(G; F) = 1$ then $H_1(K; F) = 0$, and so K is finite and $h(G) = 1$. Since K is finite it is the direct product of its Sylow subgroups, and the Sylow p -subgroup carries the p -primary homology of K . Hence if F has characteristic $p > 1$ and $H_1(K; F) = 0$ then the Sylow p -subgroup is trivial and $H_i(K; F) = 0$, for all $i \geq 1$. If F has characteristic 0 then $H_i(K; F) = 0$ for all $i \geq 1$ also. In each case, $H_i(G; F) = 0$, for all $i > 1$, and so $\beta_2(G; F) = 0$.

If $H_2(G; \mathbb{Z}) = 0$ then $\psi^{ab} - I$ is a monomorphism. Since it is a nilpotent endomorphism of K^{ab} , we must have $K^{ab} = 0$. Hence $K = 1$ and $G \cong \mathbb{Z}$. \square

In particular, if $h(G) = 1$ and T is the torsion subgroup of G then $\beta_1(T; \mathbb{F}_p) > 0$ if and only if $\beta_1(G; \mathbb{F}_p) > 1$. The fact that the torsion subgroup has non-trivial image

in the abelianization does not extend to nilpotent groups G with $h(G) > 1$, as may be seen from the groups with presentation $\langle x, y \mid [x, [x, y]] = [y, [x, y]] = [x, y]^p = 1 \rangle$.

Corollary 10. *Let G be a finitely generated nilpotent group, and let T be its torsion subgroup. Then*

- (1) $\beta_2(G; \mathbb{Q}) < \beta_1(G; \mathbb{Q})$ if and only if $h(G) = 1$ or 2 ;
- (2) if $\beta_2(G; \mathbb{F}_p) < \beta_1(G; \mathbb{F}_p)$ for some prime p then G is infinite, G has no p -torsion and $h(G) = \beta_1(G; \mathbb{F}_p) = 1$ or 2 .

Proof. If G is finite then we may assume that it is a p -group for some prime p , and then $\beta_2(G; F) = \beta_1(G; F) = 0$ for all fields of characteristic $\neq p$, while it follows from the Universal Coefficient exact sequences of §1 that $\beta_2(G; \mathbb{F}_p) \geq \beta_1(G; \mathbb{F}_p)$, since ${}_pG^{ab}$ and G^{ab}/pG^{ab} have the same dimension.

Hence we may assume that G is infinite, and so $G \cong K \rtimes_{\psi} \mathbb{Z}$, where K is a finitely generated nilpotent group and ψ is a unipotent automorphism. Let F be a field. We may use Lemma 9 to show first that $\beta_2(K; F) = 0$ and then that $\beta_1(K; F) \leq 1$. Hence $\beta_1(G; F) \leq 2$.

If $F = \mathbb{Q}$ then either K is finite and $h(G) = 1$, or $h(K) = 1$ and $h(G) = 2$. The converse is clear, since G is then a finite extension of $\mathbb{Z}^{h(G)}$.

Suppose that $F = \mathbb{F}_p$ for some prime p . If $\beta_1(G; \mathbb{F}_p) = 1$ then K is finite, so $h(G) = 1$, and $\beta_2(K; \mathbb{F}_p) = 0$, so $\beta_1(K; \mathbb{F}_p) = 0$ and K has no p -torsion. If $\beta_1(G; \mathbb{F}_p) = 2$ then $\beta_1(K; \mathbb{F}_p) = 1$ and $\beta_2(K; \mathbb{F}_p) = 0$, so $h(K) = 1$ and K has no p -torsion. Hence $h(G) = 2$ and G has no p -torsion. \square

Lemma 11. *Let G be a finitely generated nilpotent group and let $\beta = \beta_1(G; \mathbb{Q})$. Then G is homologically balanced if and only if $H_2(G; \mathbb{Z})$ is a quotient of \mathbb{Z}^{β} ; if $h(G) > 2$ then G is homologically balanced if and only if $H_2(G; \mathbb{Z}) \cong \mathbb{Z}^{\beta}$.*

Proof. Since $G^{ab} \cong \mathbb{Z}^{\beta} \oplus B$, where B is finite, the first assertion follows from the Universal Coefficient exact sequences of §1. If $h(G) > 2$ then $\beta_2(G; \mathbb{Q}) \geq \beta$, by Corollary 10, and so $H_2(G; \mathbb{Z})$ is a quotient of \mathbb{Z}^{β} if and only if $H_2(G; \mathbb{Z}) \cong \mathbb{Z}^{\beta}$. \square

6. $h = 1$: VIRTUALLY \mathbb{Z}

We include the following simple lemma as some of the observations are not explicit in our primary reference [12].

Lemma 12. *Let G be a finitely generated nilpotent group, and let T be its torsion subgroup. Then the following are equivalent*

- (1) $\beta_1(G; \mathbb{Q}) = 1$;
- (2) $h(G) = 1$;
- (3) $G/T \cong \mathbb{Z}$;
- (4) $G \cong T \rtimes_{\psi} \mathbb{Z}$, where ψ is an automorphism of T .

Proof. In each case G is clearly infinite, and so there is an epimorphism $f : G \rightarrow \mathbb{Z}$, with kernel K , say. Since G is finitely generated, so is K . If $\beta_1(G; \mathbb{Q}) = 1$ then K is finite, by Lemma 9. If $h(G) = 1$ then $h(K) = 0$, so K is again finite. In each case, $K = T$ and $G/T \cong \mathbb{Z}$. If $G/T \cong \mathbb{Z}$ and $t \in G$ represents a generator of G/T then conjugation by t defines an automorphism ψ of T , and $G \cong T \rtimes_{\psi} \mathbb{Z}$. Finally, it is clear that (4) implies each of (1) and (2). \square

We could also describe the groups considered on this lemma as the nilpotent groups which are virtually \mathbb{Z} , and as the nilpotent groups with two ends.

Theorem 13. *If π_X and π_Y are nilpotent and $\chi(X) < \chi(Y)$ then either $\pi_X \cong \mathbb{Z}$ and $\pi_Y = 1$ or $\pi_X \cong \mathbb{Z}^2$ and $\pi_Y \cong \mathbb{Z}$.*

Proof. Since $\chi(X) + \chi(Y) = 2$ and $0 \leq \chi(X) < \chi(Y)$, we must have $\chi(X) = 0$ and $\chi(Y) = 2$. Since π_X is nilpotent, X is aspherical [7, Theorem 2]. Therefore $\pi_X \cong \mathbb{Z}$ or \mathbb{Z}^2 . If $\pi_X \cong \mathbb{Z}$ then $\beta_1(Y; \mathbb{Z}) = \beta_2(X; \mathbb{Z}) = 0$ and $H_1(Y; \mathbb{Z})$ is torsion-free. Hence $\pi_Y^{ab} = 0$ and so $\pi_Y = 1$, since it is nilpotent. Similarly, if $\pi_X \cong \mathbb{Z}^2$ then $\beta_1(Y; \mathbb{Z}) = \beta_2(X; \mathbb{Z}) = 1$ and $H_1(Y; \mathbb{Z})$ is torsion-free. Hence $\pi_Y^{ab} \cong \mathbb{Z}$ and so $\pi_Y \cong \mathbb{Z}$, since it is nilpotent. \square

The pairs $(\mathbb{Z}, 1)$ and $(\mathbb{Z}^2, \mathbb{Z})$ are the pairs of fundamental groups of the complementary regions of the standard embeddings of $S^1 \times S^2$ and $S^1 \times S^1 \times S^1$ in S^4 (as the boundaries of regular neighbourhoods of the “unknotted” embeddings of S^2 and of the torus $S^1 \times S^1$). In all other cases, if π_X and π_Y are nilpotent then $\chi(X) = \chi(Y) = 1$, so $\beta_2(X) = \beta_1(X)$ and $\beta_2(Y) = \beta_1(Y)$. Since for any embedding of a 3-manifold into S^4 the groups $H_1(X; \mathbb{Z})$ and $H_1(Y; \mathbb{Z})$ have isomorphic torsion subgroups and $\beta_1(Y) = \beta_2(X)$, it follows that $\pi_X^{ab} \cong \pi_Y^{ab}$.

In the next lemma we do not assume that G is nilpotent.

Lemma 14. *Let $G \cong T \rtimes_{\psi} \mathbb{Z}$ be a homologically balanced group, where T is finite. Then $H_2(G; \mathbb{Z})$ is a finite cyclic group, and $|H_2(G; \mathbb{Z})|$ is divisible by the order of the torsion subgroup of G^{ab} .*

Proof. It is immediate from the Wang sequence for the integral homology of G as an extension of \mathbb{Z} by T that $H_2(G; \mathbb{Z})$ is finite. It is also clear that $C = \text{Cok}(\psi^{ab} - I)$ is the torsion subgroup of G^{ab} . Since T is finite, $|\text{Ker}(\psi^{ab} - I)| = |\text{Cok}(\psi^{ab} - I)|$, and so $|C|$ divides $|H_2(G; \mathbb{Z})|$.

If p is a prime then $\dim_{\mathbb{F}_p} \text{Tor}(\mathbb{F}_p, G^{ab}) = \dim_{\mathbb{F}_p} \text{Tor}(\mathbb{F}_p, C) = \beta_1(G; \mathbb{F}_p) - 1$, since $G^{ab} \cong \mathbb{Z} \oplus C$. Therefore

$$\dim_{\mathbb{F}_p} \text{Hom}(H_2(G; \mathbb{Z}), \mathbb{F}_p) = \beta_2(G; \mathbb{F}_p) - \beta_1(G; \mathbb{F}_p) + 1,$$

by the Universal Coefficient exact sequences of §1. Since G is homologically balanced, this is at most 1, for all primes p , and so $H_2(G; \mathbb{Z})$ is cyclic. \square

If G is nilpotent then $H_2(\psi) - I$ is a nilpotent endomorphism of $H_2(T; \mathbb{Z})$, and so $|H_2(G; \mathbb{Z})| = |C|$ if and only if $H_2(T; \mathbb{Z}) = 0$.

Theorem 15. *Let $G \cong T \rtimes_{\psi} \mathbb{Z}$ be a homologically balanced nilpotent group, where T is finite. Then*

- (1) G is 2-generated;
- (2) if the Sylow p -subgroup of T is abelian then it is cyclic;
- (3) if T is abelian then $G \cong \mathbb{Z}/m\mathbb{Z} \rtimes_n \mathbb{Z}$, for some $m, n \neq 0$ such that m divides a power of $n - 1$.

Proof. Let p be a prime. Then $\dim_{\mathbb{F}_p} H_1(T; \mathbb{F}_p) = \dim_{\mathbb{F}_p} \text{Tor}(T^{ab}, \mathbb{F}_p)$, since T is finite. Moreover, $\psi^{ab} - I$ and $\text{Tor}(\psi^{ab}, \mathbb{F}_p) - I$ have the same rank. Since $\text{Tor}(T^{ab}, \mathbb{F}_p)$ is a natural quotient of $H_2(T; \mathbb{F}_p)$, exactness of the Wang sequence implies that $\beta_2(G; \mathbb{F}_p) \geq 2(\beta_1(G; \mathbb{F}_p) - 1)$. Since G is homologically balanced, $\beta_1(G; \mathbb{F}_p) \leq 2$. Hence the p -torsion of G^{ab} is cyclic. Therefore $G^{ab} \cong \mathbb{Z} \oplus C$ for some finite cyclic group. Since G is nilpotent and G^{ab} is 2-generated, so is G .

The Sylow subgroups of T are characteristic, and ψ restricts to a unipotent automorphism of each such subgroup. Suppose that the Sylow p -subgroup of T is

an abelian group A . Since $H^2(A; \mathbb{F}_p) = \text{Hom}(H_2(A; \mathbb{F}_p), \mathbb{F}_p)$, the endomorphisms $H^2(\psi; \mathbb{F}_p) - I$ and $H_2(\psi; \mathbb{F}_p) - I$ have the same rank. Hence

$$\dim(\text{Ker}(H^2(\psi; \mathbb{F}_p) - I)) = \dim(\text{Ker}(H_2(\psi; \mathbb{F}_p) - I)) = \dim(\text{Cok}(H_2(\psi; \mathbb{F}_p) - I)) \leq 1.$$

Hence A must be cyclic, by Corollary 8.

It follows immediately that if T is abelian then it is a direct product of cyclic groups of relatively prime orders, and so is cyclic, of order m , say. Hence $G \cong \mathbb{Z}/m\mathbb{Z} \rtimes_n \mathbb{Z}$, for some n such that $(m, n) = 1$. Such a semidirect product is nilpotent if and only if m divides some power of $n - 1$. \square

If T is a homologically balanced, 2-generator nilpotent group and ψ is a unipotent automorphism of T such that $H_1(\psi; \mathbb{F}_p) \neq I$ for any p dividing $|T|$ then exactness of the Wang sequence implies that $T \rtimes_\psi \mathbb{Z}$ is homologically balanced.

Every semidirect product $\mathbb{Z}/m\mathbb{Z} \rtimes_n \mathbb{Z}$ has a balanced presentation

$$\langle a, t \mid a^m = 1, tat^{-1} = a^n \rangle.$$

If $(n - 1, \ell) = (n - 1, m)$ then $\mathbb{Z}/\ell\mathbb{Z} \rtimes_n \mathbb{Z}$ and $\mathbb{Z}/m\mathbb{Z} \rtimes_n \mathbb{Z}$ have isomorphic abelianizations, and so every such pair of groups can be realized by an embedding [10].

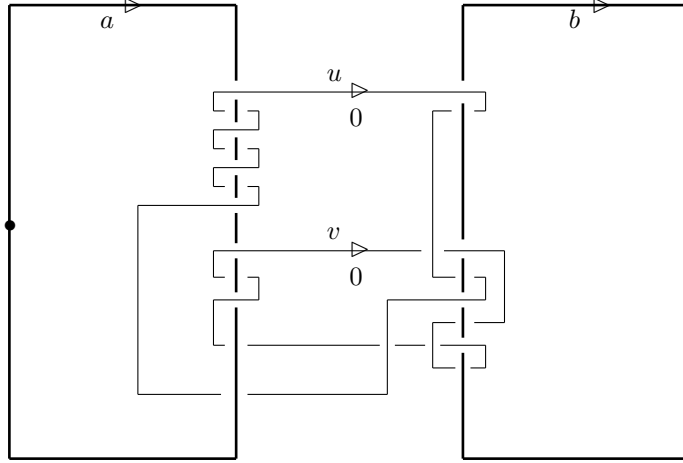


Figure 1

The simplest non-abelian nilpotent example corresponds to the choice $\ell = 2, m = 4$ and $n = -1$. One group is $\mathbb{Z}/4\mathbb{Z} \rtimes_{-1} \mathbb{Z}$, and the other is its abelianization $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. We shall give an explicit construction of an embedding realizing this pair of groups (corresponding to $\ell = 2, m = 4$ and $n = -1$). Let M be the 3-manifold obtained by 0-framed surgery on the 4-component link L depicted in Figure 1. This link is partitioned into two trivial sublinks, one of which is dotted. We modify one hemisphere of S^4 by deleting a pair of trivial 2-handles with boundaries the dotted loops and attaching 0-framed 2-handles along the other loops. This gives a region $X \subset S^4$ with $\partial X = M$. The complement $Y = \overline{S^4} \setminus X$ then may be obtained from the other hemisphere by swapping the roles of the dotted and undotted loops. The fundamental groups of X and Y have presentations $\langle a, b \mid U = V = 1 \rangle$ and $\langle u, v \mid A = B = 1 \rangle$, where the words $A = u^4 v^2$, $B = vuv^{-1}u^{-1}$, $U = a^4$ and

$V = b^{-1}aba$, are easily read from the diagram. Thus the embedding of M is nilpotent, with $\pi_X \cong \mathbb{Z}/4\mathbb{Z} \rtimes_{-1} \mathbb{Z}$, and $\pi_Y \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

The simplest examples with T non-abelian are the groups $Q(8k) \rtimes \mathbb{Z}$, with the balanced presentations $\langle t, x, y \mid x^{2k} = y^2, tx = xt, tyt^{-1} = xy \rangle$, which simplify to

$$\langle t, y \mid [t, y]^{2k} = y^2, [t, [t, y]] = 1 \rangle.$$

Let $m = p^s$, where p is a prime and $s \geq 1$, and let G be the group with presentation

$$\langle t, x, y \mid txt^{-1} = y, tyt^{-1} = x^{-1}y^2, yxy^{-1} = x^{m+1} \rangle.$$

If we conjugate the final relation with t to get the relation $x^{-1}yx = y^{m+1}$ then we see that the torsion subgroup T has presentation $\langle x, y \mid x^m = y^m, yxy^{-1} = x^{m+1} \rangle$, and so is one of the metacyclic p -groups mentioned at the end of §2. Moreover, G is nilpotent, and $\zeta G = \langle x^m \rangle$ and $G' = \langle x^m, x^{-1}y \rangle$ are abelian. Hence G is metabelian. Each of the groups that we have described here is 2-generated and its torsion subgroup is homologically balanced.

7. METABELIAN NILPOTENT GROUPS WITH HIRSCH LENGTH > 1

All known examples of nilpotent groups with balanced presentations and Hirsch length $h > 1$ are torsion-free. We have not yet been able to show that this must be so. However, if such a group is also metabelian, but not \mathbb{Z}^3 , then $h(G) \leq 4$ and $\beta_1(G; \mathbb{Q}) = 2$ [8, Theorems 7 and 15]. Our main result implies that there are just three such groups with $G/G' \cong \mathbb{Z}^2$. The argument again involves finding normal subgroups with “large enough” homology to affect the Betti numbers of the group. We develop a number of lemmas to this end.

Lemma 16. *Let G be a finitely generated nilpotent group, and let T be its torsion subgroup. Then the following are equivalent*

- (1) $\beta_1(G; \mathbb{Q}) = 2$ and $\beta_2(G; \mathbb{Q}) = 1$;
- (2) $h(G) = 2$;
- (3) $G/T \cong \mathbb{Z}^2$.

If these conditions hold and G is homologically balanced then $H_2(G; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/e\mathbb{Z}$, for some $e \geq 1$.

Proof. If (1) holds then $h(G) \geq 2$, and so $h(G) = 2$, by the corollary to Lemma 9. It is easy to see that (2) and (3) are equivalent, and imply (1). The final assertion follows from Corollary 10 and Lemma 11. \square

We could also describe the groups considered in this lemma as the nilpotent groups which are virtually \mathbb{Z}^2 .

In this section the Lyndon-Hochschild-Serre spectral sequences for the homology and cohomology of a group which is an extension of \mathbb{Z}^2 by a normal subgroup shall largely replace the Wang sequences used above.

Lemma 17. *Let F be a field and A be a finite dimensional $F[\mathbb{Z}^2]$ -module, and let $b_i = \dim_F H_i(\mathbb{Z}^2; A)$, for $i \geq 0$. Then $b_2 = b_0$ and $b_1 = b_0 + b_2 = 2b_0$.*

Proof. We may compute $H_i(\mathbb{Z}^2; A) = \text{Tor}_i^{F[\mathbb{Z}^2]}(F, A)$ from the complex

$$0 \rightarrow A \rightarrow A^2 \rightarrow A \rightarrow 0,$$

in which the differentials are $\partial^1 = \begin{bmatrix} (x-1)id_A \\ (y-1)id_A \end{bmatrix}$, and $\partial^2 = [(y-1)id_A, (1-x)id_A]$ where $\{x, y\}$ is a basis for \mathbb{Z}^2 . Since the matrix for ∂^2 is the transpose of that for ∂^1

(up to a change of sign in the second block), they have the same rank. Hence $b_2 = \dim_F \text{Ker}(\partial_2) = \dim_F \text{Cok}(\partial_1) = b_0$. The final assertion follows since $b_0 - b_1 + b_2 = 1 - 2 + 1 = 0$ is the Euler characteristic of the complex. \square

The modules $H_2(\mathbb{Z}^2; A)$ and $H_0(\mathbb{Z}^2; A)$ are the submodule of fixed points and the coinvariant quotient modules of the \mathbb{Z}^2 -action, respectively. Minor adjustments give similar results for $\dim_F H^j(\mathbb{Z}^2; A)$. (We may also use Poincaré duality for \mathbb{Z}^2 to relate homology and cohomology.)

Recall that if K is a normal subgroup of a group G then conjugation in G induces a natural action of G/K on the homology and cohomology of K .

Lemma 18. *Let G be a finitely generated nilpotent group with a normal subgroup K such that $G/K \cong \mathbb{Z}^2$, and suppose that $\beta_2(G; F) \leq \beta_1(G; F)$ for some field F . Then $\dim_F H^2(K; F)^{G/K} \leq 1$.*

Proof. We note first that $\beta_1(G; F) = 2$ or 3 , since $\beta_2(G; F) \leq \beta_1(G; F)$ [11, Theorem 2.7]. We may assume that $A = H^1(K; F) \neq 0$, since G is nilpotent. Let $N = G/K$ and $b_i = \dim_F H^i(N; A)$. The LHS spectral sequence for cohomology with coefficients F for G as an extension of N by K gives two exact sequences

$$0 \rightarrow H^1(N; F) \rightarrow H^1(G; F) \rightarrow A^N \xrightarrow{d_2^{0,1}} H^2(N; F) \rightarrow H^2(G; F) \rightarrow J \rightarrow 0$$

and

$$0 \rightarrow H^1(N; A) \rightarrow J \rightarrow H^2(K; F)^N \xrightarrow{d_2^{0,2}} H^2(N; A).$$

The first sequence gives $\dim_F J \leq \beta_2(G; F)$. Then $b_1 = b_0 + b_2 = 2b_0$, by Lemma 17, and $b_0 > 0$, since $A \neq 0$. Hence $b_0 - b_1 < 0$, and so the second sequence gives $\dim_F H^2(K; F)^N \leq \beta_2(G; F) + b_0 - b_1 \leq \beta_2(G; F) - 1$. In particular, $\dim_F H^2(K; F)^N \leq 1$ if $\beta_2(G; F) = 2$.

If $\beta_2(G; F) = 3$ then $\beta_1(G; F) = 3$ also, by Corollary 10 and so $\dim_F \text{Ker}(d_2^{0,1}) = 1$. If $d_2^{0,1} \neq 0$ then $b_0 = 2$ and so $b_1 = 4$, by Lemma 17. But then $\beta_2(G; F) \geq 4$. Therefore $d_2^{0,1} = 0$, and so $b_0 = 1$. Hence $b_1 = 2$ and $b_2 = 1$, and $d_2^{0,2}$ is a monomorphism. Hence we again have $\dim_F H^2(K; F)^N \leq 1$. \square

In particular, $H^2(K; F)$ is not canonically subsplit.

A parallel argument using the LHS spectral sequence for homology shows that $\dim_F H_0(G/K; H_2(K; F)) \leq 1$.

Lemma 19. *Let P be a non-trivial finite p -group and $K \cong \mathbb{Z} \times P$. Then $H^2(K; \mathbb{F}_p)$ is canonically subsplit.*

Proof. We shall use the Universal Coefficient exact sequence for cohomology given in §1. The projection of K onto $K/P \cong \mathbb{Z}$ determines a class $\eta \in H^1(K; \mathbb{F}_p) = \text{Hom}(K, \mathbb{F}_p)$ (up to sign), and cup product with η maps $H^1(K; \mathbb{F}_p)$ non-trivially to $H^2(K; \mathbb{F}_p)$, by the Künneth Theorem. The restriction from $\text{Ext}(K^{ab}, \mathbb{F}_p)$ to $\text{Ext}(P^{ab}, \mathbb{F}_p)$ is an isomorphism, and so $\text{Ext}(K^{ab}, \mathbb{F}_p)$ and $\eta \cup H^1(K; \mathbb{F}_p)$ have trivial intersection. Hence $\text{Ext}(K^{ab}, \mathbb{F}_p) \oplus (\eta \cup H^1(K; \mathbb{F}_p))$ is a subspace of $H^2(K; \mathbb{F}_p)$, and the summands are invariant under the action of automorphisms of K , by the naturality of the Universal Coefficient Theorem. The summands are non-trivial, since $P \neq 1$. \square

The next four lemmas (leading up to Theorem 24) consider nilpotent groups which are extensions of \mathbb{Z}^2 by finite normal subgroups.

Lemma 20. *Let G be a finitely generated nilpotent group, and let T be its torsion subgroup. Let P be a non-trivial Sylow p -subgroup of T and let $\gamma_p : G \rightarrow \text{Out}(P)$ be the homomorphism determined by conjugation in G . If $G/T \cong \mathbb{Z}^2$ and the image of γ_p is cyclic then $\beta_2(G; \mathbb{F}_p) > \beta_1(G; \mathbb{F}_p)$.*

Proof. We may write $G \cong K \rtimes_{\psi} \mathbb{Z}$, where ψ is a unipotent automorphism of K , and K is in turn an extension of \mathbb{Z} by T . Let P be the Sylow p -subgroup of T , and let N be the product of the other Sylow subgroups of T . Since the Sylow subgroups of T are characteristic, conjugation in G determines a homomorphism $\gamma_p : G \rightarrow \text{Out}(P)$. Moreover, N is normal in G , and the projection of G onto G/N induces isomorphisms on homology and cohomology with coefficients \mathbb{F}_p . Hence we may assume that $N = 1$ and so $T = P$ is a non-trivial p -group.

If the image of γ_p is cyclic then γ_P factors through an epimorphism $f : G \rightarrow \mathbb{Z}$, with kernel $K \cong \mathbb{Z} \times P$. Since $H^2(K; \mathbb{F}_p)$ has a subspace which is the direct sum of non-trivial canonical summands, by Lemma 19, $\dim_{\mathbb{F}_p} \text{Ker}(H^2(\psi; \mathbb{F}_p) - I) > 1$ (as in Lemma 6). The result now follows from Lemma 9. \square

Thus the group with presentation $\langle x, y \mid [x, [x, y]] = [y, [x, y]] = [x, y]^p = 1 \rangle$ mentioned near the end of §1 above does not have a balanced presentation. Similarly, no nilpotent extension of \mathbb{Z}^2 by $Q(8)$ can have a balanced presentation, since the abelian subgroups of $\text{Out}(Q(8)) \cong \mathcal{S}_3$ are cyclic.

If p is an odd prime and C is a cyclic p -group then $\text{Aut}(C)$ is cyclic, and so Lemma 20 may apply. However, dealing with 2-torsion again requires more effort.

Lemma 21. *Let G be a finitely generated nilpotent group, and let T be its torsion subgroup. If $G/T \cong \mathbb{Z}^2$ and the Sylow 2-subgroup of T is a nontrivial cyclic group then $\beta_2(G; \mathbb{F}_2) > \beta_1(G; \mathbb{F}_2)$.*

Proof. We may factor out the maximal odd-order subgroup of T without changing the \mathbb{F}_2 -homology, and so we may assume that $T \cong \mathbb{Z}/k\mathbb{Z}$, where $k = 2^n$, for some $n \geq 1$. We may also assume that the action of G on T by conjugation does not factor through a cyclic group, by Lemma 20, and so $k \geq 8$. Let U be the subgroup of $(\mathbb{Z}/k\mathbb{Z})^\times$ represented by integers $\equiv 1 \pmod{4}$. Then $\text{Aut}(\mathbb{Z}/k\mathbb{Z}) \cong \{\pm 1\} \times U$. It is easily verified that noncyclic subgroups of $\text{Aut}(\mathbb{Z}/k\mathbb{Z})$ have $\{\pm 1\}$ as a direct factor, and so G has a presentation

$$\langle x, y, z \mid [x, y] = z^f, z^k = 1, xzx^{-1} = z^{-1}, yzy^{-1} = z^\ell \rangle,$$

where f divides k , $1 < \ell < k$ and $\ell \equiv 1 \pmod{4}$. Let m be a multiplicative inverse for $\ell \pmod{k}$, so that $1 < m < k$ and $m\ell = wk + 1$ for some $w \in \mathbb{Z}$. Note that $\beta_1(G; \mathbb{F}_2) = 2$ if $f = 1$ and is 3 if $f > 1$.

The ring $\mathbb{Z}[G]$ is a twisted polynomial extension of $\mathbb{Z}[\mathbb{Z}/k\mathbb{Z}] = \mathbb{Z}[z]/(z^k - 1)$, and so is noetherian. We may assume each monomial is normalized in alphabetical order: $x^h y^i z^j$, for exponents $h, i \in \mathbb{Z}$ and $0 \leq j < k$. Let $\nu = \sum_{i=0}^{k-1} z^i$ be the norm element for $\mathbb{Z}[\mathbb{Z}/k\mathbb{Z}]$. Then $z\nu = \nu$, so $\nu^2 = k\nu$, and ν is central in $\mathbb{Z}[G]$. We shall use the fact that if $\gamma, \delta \in \mathbb{Z}[G]$ are such that $\gamma\nu = 0$ and $\delta(z - 1) = 0$ then $\gamma = \gamma'(z - 1)$ and $\delta = \delta'\nu$, for some $\gamma', \delta' \in \mathbb{Z}[G]$. On the other hand, non-zero terms not involving z are not zero-divisors in $\mathbb{Z}[G]$.

The augmentation module \mathbb{Z} has a Fox-Lyndon partial resolution

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 = \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0,$$

where $\varepsilon : \mathbb{Z}[G] \rightarrow \mathbb{Z}$ is the augmentation homomorphism, $C_1 \cong \mathbb{Z}[G]^3$ has basis $\{e_x, e_y, e_z\}$ corresponding to the generators and $C_2 \cong \mathbb{Z}[G]^4$ has basis $\{r, s, t, u\}$ corresponding to the relators $r = z^f y x y^{-1} x^{-1}$, $s = z^k$, $t = z x z x^{-1}$ and $u = z^\ell y z^{-1} y^{-1}$. The differentials are given by

$$\begin{aligned} \partial_1(e_x) &= x - 1, & \partial_1(e_y) &= y - 1 & \text{and} & \partial_1(e_z) &= z - 1; & \text{and} \\ \partial_2(r) &= (z^f y - 1)e_x + (z^f - x)e_y + (\sum_{i=0}^{f-1} z^i)e_z, & \partial_2(s) &= \nu e_z, \\ \partial_2(t) &= (z - 1)e_x + (1 + zx)e_z & \text{and} & \partial_2(u) &= (z^\ell - 1)e_y + (\sum_{j=0}^{\ell-1} z^j - y)e_z. \end{aligned}$$

We may choose a homomorphism $\partial_3 : C_3 \rightarrow C_2$ with domain C_3 a free $\mathbb{Z}[G]$ -module and image $\text{Ker}(\partial_2)$, which extends the resolution one step to the left. (We may assume that C_3 is finitely generated, since $\mathbb{Z}[G]$ is noetherian.) It is clear from the Fox-Lyndon partial resolution that $\dim_{\mathbb{F}_2} \text{Ker}(\mathbb{F}_2 \otimes_{\mathbb{Z}[G]} \partial_2) = \beta_1(G; \mathbb{F}_2) + 1$. We shall show that $\mathbb{F}_2 \otimes_{\mathbb{Z}[G]} \partial_3 = 0$, and so $\beta_2(G; \mathbb{F}_2) = \beta_1(G; \mathbb{F}_2) + 1$.

Let $\varepsilon_2 : \mathbb{Z}[G] \rightarrow \mathbb{F}_2$ be the *mod* (2) reduction of ε . Since $\text{Im}(\partial_3) = \text{Ker}(\partial_2)$, it shall suffice to show that if

$$\partial_2(ar + bs + ct + du) = 0,$$

for some $a, b, c, d \in \mathbb{Z}[G]$ then $\varepsilon_2(a) = \varepsilon_2(b) = \varepsilon_2(c) = \varepsilon_2(d) = 0$.

The coefficients a, b, c, d must satisfy the three equations

$$\begin{aligned} a(z^f y - 1) + c(z - 1) &= 0, \\ a(z^f - x) + d(z^\ell - 1) &= 0 \end{aligned}$$

and

$$a(\sum_{i=0}^{f-1} z^i) + b\nu + c(zx + 1) + d(\sum_{j=0}^{\ell-1} z^j - y) = 0.$$

Multiplying the first of these equations by ν gives $af\nu(y - 1) = 0$. Hence $a\nu = 0$ and so $a = A(z - 1)$, for some $A \in \mathbb{Z}[G]$ not involving z . The first equation becomes

$$A(z - 1)(z^f y - 1) + c(z - 1) = [A(yz^{fm}(\sum_{j=0}^{m-1} z^j) - 1) + c](z - 1) = 0,$$

and so $c = -A(yz^{fm}(\sum_{j=0}^{m-1} z^j) - 1) + C\nu$, for some $C \in \mathbb{Z}[G]$ not involving z . Similarly, the second equation becomes

$$A(z - 1)(z^f - x) + d(z^\ell - 1) = A(zx + z^f)(z^{m\ell} - 1) + d(z^\ell - 1) = 0,$$

and so $d = -A(zx + z^f)(\sum_{j=0}^{m-1} z^{j\ell}) + D\nu$, for some $D \in \mathbb{Z}[G]$ not involving z . At this point it is already clear that $\varepsilon_2(a) = \varepsilon_2(c) = \varepsilon_2(d) = 0$.

Multiplying the third equation by ν gives

$$kb\nu + c\nu(x + 1) + d\nu(\ell - y) = 0.$$

Rearranged and written out in full, this becomes

$$kb\nu = (A(yx - 1) - Ck)(x + 1)\nu + (A(x + 1)m - Dk)(\ell - y)\nu.$$

Since $yx = z^{-f}xy = xzy = xyz^{fm}$ we have $yx\nu = xy\nu$ and so this simplifies to

$$kb\nu = (A(m\ell - 1)(x + 1) - kC(x + 1) - kD(\ell - y))\nu.$$

Write $b = b_1 + B(z - 1)$, where b_1 does not involve z . Then $b\nu = b_1\nu$. Since the terms b_1, A, C and D do not involve z , and since $m\ell - 1 = wk$, we get

$$kb_1 = k(Aw(x + 1) - C(x + 1) - D(\ell - y)).$$

We may solve for b_1 , and so

$$b = b_1 + B(z - 1) = wA(x - 1) + B(z - 1) - C(x + 1) - D(\ell - y).$$

Hence $\varepsilon_2(b) = 0$ also, so $\mathbb{F}_2 \otimes_{\mathbb{Z}[G]} \partial_3 = 0$ and thus $\beta_2(G; \mathbb{F}_2) = \beta_1(G; \mathbb{F}_2) + 1$. \square

Lemma 22. *Let G be a finitely generated nilpotent group, and let T be its torsion subgroup. If $G/T \cong \mathbb{Z}^2$ and the Sylow p -subgroup of T is abelian and non-trivial then $\beta_2(G; \mathbb{F}_p) > \beta_1(G; \mathbb{F}_p)$.*

Proof. Let N be the product of all the Sylow p' -subgroups of T with $p' \neq p$, and let A be the image of T in $\overline{G} = G/N$. Then $\beta_i(\overline{G}; \mathbb{F}_p) = \beta_i(G; \mathbb{F}_p)$, for all i . The Sylow p -subgroup of T projects isomorphically onto A , and $\overline{G}/A \cong \mathbb{Z}^2$. If $\beta_2(G; \mathbb{F}_p) \leq \beta_1(G; \mathbb{F}_p)$ then $\dim_{\mathbb{F}_p} H^2(A; \mathbb{F}_p)^{G/A} \leq 1$, by Lemma 18. Hence A is cyclic, by Lemmas 6 and 7. If $p = 2$ the result follows from Lemma 21, while if p is odd it follows from Lemma 20, since the automorphism group of a cyclic group of odd p -power order is cyclic. \square

For the next result we need an analogue of Lemma 19.

Lemma 23. *Let $K \cong T \rtimes \mathbb{Z}^2$, where T is a finite p -group, and $T \not\leq K'$. Then $H^2(K; \mathbb{F}_p)$ is canonically subsplit.*

Proof. Let $\alpha : K \rightarrow \mathbb{Z}^2$ be the canonical epimorphism. Since α splits, $H^2(\alpha; \mathbb{F}_p)$ is a monomorphism. The other hypotheses imply $\text{Ext}(K^{ab}, \mathbb{F}_p) \neq 0$. Hence $\text{Ext}(K^{ab}, \mathbb{F}_p) \oplus \text{Im}(H^2(\alpha; \mathbb{F}_p))$ is a subspace of $H^2(K; \mathbb{F}_p)$ with the desired properties. \square

If G is a homologically balanced, metabelian nilpotent group then either $G \cong \mathbb{Z}^3$ or $\beta_1(G; \mathbb{Q}) \leq 2$ and $h(G) \leq 4$. In the latter case the torsion-free quotient G/T is either free abelian of rank ≤ 2 , or is a Nil^3 -group Γ_q with presentation

$$\langle x, y, z \mid [x, y] = z^q, \quad xz = zx, \quad yz = zy \rangle,$$

or is the Nil^4 -group Ω with presentation

$$\langle t, u \mid [t, [t, [t, u]]] = [u, [t, u]] = 1 \rangle.$$

(See [8, Corollary 8 and Theorems 10 and 15].)

Theorem 24. *Let G be a homologically balanced nilpotent group with $\beta_1(G; \mathbb{Q}) = 2$, and let T be its torsion subgroup. Then*

- (1) *if $h(G) = 2$ and T is abelian then $G \cong \mathbb{Z}^2$;*
- (2) *if $h(G) = 3$ and the outer action $: G \rightarrow \text{Out}(T)$ determined by conjugation in G factors through \mathbb{Z}^2 then $G \cong \Gamma_q$, for some $q \geq 1$;*
- (3) *if $h(G) = 4$ and G has an abelian normal subgroup A with $G/AT \cong \mathbb{Z}^2$ then $G \cong \Omega$.*

Proof. Let K be the preimage in G of the torsion subgroup of G^{ab} . Then $TG' \leq K$, $G/K \cong \mathbb{Z}^2$, and K/G' is (finite) cyclic, since $\beta_1(G; \mathbb{F}_p) \leq 3$ for all primes p .

If $h(G) = 2$ then $K = T$ and $G/T \cong \mathbb{Z}^2$. Hence if T is abelian then $T = 1$, by Lemma 22, and so $G \cong \mathbb{Z}^2$.

If $h(G) = 3$ and the outer action $: G \rightarrow \text{Out}(T)$ determined by conjugation in G factors through \mathbb{Z}^2 then $K \cong T \times \mathbb{Z}$. Thus if $T \neq 1$ then $\beta_2(G; \mathbb{F}_p) > \beta_1(G; \mathbb{F}_p)$, for any prime p dividing $|T|$, by Lemmas 18 and 19. This contradicts the hypothesis that G has a balanced presentation.

If $h(G) = 4$ and G has an abelian normal subgroup A with such that $G/AT \cong \mathbb{Z}^2$ then $K = AT$, and $\overline{A} = A/A \cap T \cong \mathbb{Z}^2$. Since K is nilpotent, the action of the finite group $T/A \cap T$ on \overline{A} is trivial. Hence $K/A \cap T \cong \overline{A} \times (T/A \cap T)$, and so

$K \cong T \rtimes \mathbb{Z}^2$. Moreover, if $T \neq 1$ then $T \not\leq K'$. Lemmas 18 and 23 (together with Lemmas 6 and 7) then give a similar contradiction.

In parts (2) and (3) the group G is torsion-free, and so must be one of the known examples given above. \square

Imposing a stronger constraint gives a clearer statement.

Corollary 25. *If G is a homologically balanced nilpotent group with an abelian normal subgroup A such that $G/A \cong \mathbb{Z}^2$ then $G \cong \mathbb{Z}^2, \Gamma_q$ (for $q \geq 1$) or Ω . \square*

Note that the second hypotheses in parts (2) and (3) are not by themselves equivalent to assuming that G is metabelian, while the hypothesis in the corollary is somewhat stronger.

The above work leaves open the following questions, for G a nilpotent group with torsion subgroup T and a balanced presentation.

- (1) if $h(G) = 1$ is $H_2(T; \mathbb{Z}) = 0$?
- (2) If $h(G) = 2, 3$ or 4 and G is metabelian, is $T = 1$?
- (3) more ambitiously, if $h(G) > 1$ is $T = 1$?

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