STOCHASTIC LANDAU-LIFSHITZ-GILBERT EQUATIONS FOR FRUSTRATED MAGNETS UNDER FLUCTUATING CURRENTS

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ABSTRACT. We examine a stochastic Landau-Lifshitz-Gilbert equation for a frustrated ferromagnet with competing first and second order exchange interactions exposed to deterministic and random spin transfer torques in form of transport noise. We prove existence and pathwise uniqueness of weak martingale solutions in the energy space. The result ensures the persistence of topological patterns, occurring in such magnetic systems, under the influence of a fluctuating spin current.

CONTENTS

Date: June 28, 2023.
2020 *Mathematics Subject Classification.* 35D30, 35K45, 35K55, 35Q56, 35Q60, 60H15.
Key words and phrases. Landau–Lifshitz equations, stochastic partial differential equations, gradient noise.
This work was supported by the Australian Research Council Project DP200101866 and the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) Project 442047500 - SFB 1481. CM is grateful for the support and hospitality of the Sydney Mathematical Research Institute (SMRI). .

1. INTRODUCTION

Magnetic systems with higher order exchange interactions are known to host topological pattern in form of skyrmions and hopfions in dimension $d = 2,3$, respectively, occurring as isolated solitons or condensed in a regular lattice [23]. This can be seen as an emergent phenomenon arising from competing exchange interactions beyond nearest neighbours on the atomic scale, including the case of geometric frustration with alternating (anti-)ferromagnetic coupling [1]. The controlled manipulation and transport of such structures by means of external currents is at the core of possible applications in future information technologies and a key challenge for the mathematical theory [21]. A decisive aspect on the level of such nanostructure is stability with respect to fluctuations coming from external sources. The mathematical description of random effects in magnetism is generally based on stochastic Landau-Lifshitz-Gilbert equations (SLLG) for a governing micromagnetic energy and a noise. The existence of weak martingale solutions of the SLLG with heat-bath noise in dimension $d \leq 3$ and pathwise uniqueness in dimension $d = 1$ have been studied in [2, 3, 13].

The conventional mathematical theory of Landau-Lifshitz-Gilbert equations (LLG) is based on the Dirichlet energy arising from classical Heisenberg interaction of neighbouring spins. A crucial mathematical feature is the possibility of finite time singularities in spatial dimensions greater than one. Global regularity is only expected for small initial date, so that weak solution concepts become unavoidable. In the energy critical dimension $d = 2$, the blow-up scenario of dissipative harmonic flows including LLG is well-understood [12, 14, 22]. The bubbling analysis singles out a suitable notion of energy decreasing weak solutions, so-called Struwe solutions, that are unique in this class. Corresponding existence and uniqueness results in the stochastic cases are just starting to emerge [15]. The occurrence of topological singularities, however, goes hand in hand with the collapse of topological patterns. Bubbling analysis extend to models for chiral skyrmions [8, 16], which are lower order perturbations of the classical theory, while the prediction of global regularity versus finite time blow-up remains a major open problem.

Higher order bi-harmonic exchange interaction exclude topological singularities in spatial dimensions $d \leq 3$ by means of an infinite energy barrier. Global existence of topological patterns coupled to a Vlasov-Maxwell equation for the electron distributions function of an interacting current has been confirmed in [7]. In this work, we develop a stochastic framework for a LLG system with higher order exchange interactions and fluctuating currents and provide conditions under which topological patterns will persists almost surely. The models under consideration are based on interaction energies of the form

$$E(M) = \frac{1}{2}\int_{\mathbb{R}^d} \left(|\Delta M|^2 + \lambda|\nabla M|^2 + h|M - e_3|^2\right)\,\mathrm{d}x$$

for magnetization fields $M : \mathbb{R}^d \to \mathbb{S}^2 \subset \mathbb{R}^3$ where $d = 2,3$ with $\lambda \in \mathbb{R}$, $h \geq 0$ and $e_3 := (0,0,1)$. By scaling we can assume $\lambda = \pm 1$. The case we are focusing on is $\lambda = -1$ corresponding to a frustrated magnet. In this case the usual stability condition for the ferromagnetic state $m = e_3$ is $h > 1/4$ which implies coercivity $E(M) \gtrsim \int |\nabla^2 M|^2 + |M - e_3|^2$. The unperturbed Landau-Lifshitz-Gilbert equation reads

$$\partial_t M = -M \times \boldsymbol{H}(M) - \alpha M \times M \times \boldsymbol{H}(M)$$

where $\alpha > 0$ is the damping constant and

$$\boldsymbol{H}(M) = -\left[\Delta^2 M + \Delta M - h e_3\right]$$

the effective field, i.e., minus the L^2 energy gradient. The dissipative nature implies a uniform bound of $M - e_3$ in H^2 in terms of initial conditions. Here we are interested in the LLG dynamics driven by a fluctuating current. The mathematical description of such non-variational effects requires careful considerations about the specific formulation of dissipation in the style of Landau-Lifshitz or Gilbert, respectively. The so-called adiabatic spin transfer torque induced by a spin velocity field v is introduced by adding the convection term $(v \cdot \nabla)M$ to the micromagnetic torque $M \times \boldsymbol{H}(M)$ in LLG (note that this gives rise to two distinct terms). The phenomenological non-adiabatic spin transfer torque is induced by subtracting a perpendicular counterpart $\beta M \times (v \cdot \nabla)M$. The full torque becomes

$$\boldsymbol{\tau}(M) = M \times [\boldsymbol{H}(M) - \beta(v \cdot \nabla)M] + (v \cdot \nabla)M$$

so that

$$\partial_t M = -\boldsymbol{\tau}(M) - \alpha M \times \boldsymbol{\tau}(M).$$

In the special case $\alpha = \beta$, referred to as the Galilean invariant case, the interaction reduces, in the above formulation of LLG, to a single term $(1+\alpha^2)(v\cdot\nabla)M$. It is known that in general the non-Galilean model is required for an adequate description of current driven magnetic microstructures, e.g. domain walls and vortices [17, 18]. We decompose the fluctuating spin current $v_{\text{fluc}} = v + \xi$ into a deterministic temporally homogeneous part v and a space-time noise $\xi = \dot{W}$ that is formally the generalised time derivative (in the framework of Stratonovich calculus) of a Wiener process W with values in a vectorial Sobolev space to be specified below. We shall treat the deterministic part of the spin transfer torque in its general form including adiabatic and non-adiabatic terms without any balancing conditions. For the random part we focus on the Galilean invariant case, i.e., we drop stochastic terms of the form $M \times (\xi\cdot\nabla)M$ and leave only a linear transport term $(\xi\cdot\nabla)M$, which has been studied in models of stochastic Euler and Navier-Stokes equations [4, 20]. Therefore, the stochastic equation that we study in this paper is in the form

$$\begin{aligned}\partial_t M &= -M\times \boldsymbol{H}(M) - \alpha M\times(M\times \boldsymbol{H}(M)) \\ &\quad -(1+\alpha\beta)(v\cdot\nabla)M - (\alpha-\beta)M\times(v\cdot\nabla)M - (\xi\cdot\nabla)M,\end{aligned} \tag{1.1}$$

where the noise is understood in the Stratonovich sense to ensure M takes values in the unit sphere (norm constraint). This equation is written formally later in (2.3) along with a precise definition of W.

For (1.1), the corresponding Stratonovich correction is in the form $\frac{1}{2}(\xi\cdot\nabla)^2 M$, which has a lower order than the leading bi-Laplacian term that arises from the dissipation

$$\alpha M \times \left(M\times\Delta^2 M\right) = -\alpha\Delta^2 M + \alpha\left\langle M, \Delta^2 M\right\rangle M$$

under the norm constraint. Intuitively, in view of semigroup theory for mild solutions, this suggests that the noise does not need to be small or of a scale comparable to α to guarantee existence of a solution. This is indeed the case, since in energy estimates, Itô's formula leads to cancellations between Stratonovich and Itô corrections which reduce the order of norm required to estimate these noise-related terms. They can be bounded by the energy norm of M with boundedness but not smallness conditions of the noise.

Small noises are of an independent interest in literature. Recent studies [9, 11] showed that certain nonlinear PDEs perturbed by spatially divergence free linear transport noises (satisfying suitable L^∞ and L^2 conditions), converge weakly to parabolic deterministic equations. In these works, the noise (martingale) part vanishes but Stratonovich correction (in the form ΔM) stays and acts as an additional dissipation in the limit under suitable scaling, which can delay blow-ups. It is not known whether a similar result holds for the conventional LLG model (without bi-harmonic interaction) or for our LLG model in higher dimensions.

In addition, to the best of our knowledge, nonlinear transport noise of the form $M\times(\xi\cdot\nabla)M$ (which we dropped) has not been widely explored. For our model, it is inconclusive whether cancellations of highest-order terms can be achieved for Stratonovich and Itô corrections related to this kind of nonlinear noise. Those highest-order terms consist of cross products of mixed derivatives, which seem to require H^3-estimates in dimension $d\geq 2$, i.e., the energy norm is not sufficient. It is therefore another open question regarding the existence of solution when the random part is also in the general non-Galilean form.

In this paper, we show that under certain (spatial) regularity of the noise, there exists a pathwise unique solution of a stochastic LLG equation in the form (1.1) for frustrated magnets, with H^2-moment estimates. We formulate the formal stochastic LLG equation (2.3) in Section 2.1 and provide the main results (Theorem 2.1 and Corollary 2.2) in Section 2.2. The rest of the paper is devoted to the proof of Theorem 2.1. We first construct an approximating equation in Section 3.1 using standard mollifiers and a suitable cut-off function. The latter allows us to estimate certain nonlinear terms in the absence of norm constraint, i.e., when the approximation does not necessarily take values in the unit sphere. This becomes clear as we derive H^2-uniform estimates of the approximation in Section 3.2, where the norm constraint is narrowly violated due to mollifications. Applying Skorohod theorem and compactness embedding results, we deduce strong (resp., weak) convergence in a weighted H^1 (resp., H^2) space in Section 4. Subsequently, we verify that the limit takes values in the unit sphere (Section 4.1) and prove convergences of the drift and the diffusion terms of the equation separately (Section 4.3). In Section 5, we show that the limit is indeed a pathwise unique solution of our stochastic LLG, concluding the proof. For the ease of reading, we collect in Appendix A some preliminary estimates used for Section 3.2. Similar arguments hold for the equation on the torus, for which we give a brief description in Appendix B.

2. Problem formulation and results

2.1. **Notation and the equation.** We denote by $\mathbb{S}^2 \subset \mathbb{R}^3$ the unit sphere. Let $d = 2,3$. Let $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{D}(\mathbb{R}^d)$ denote the Schwartz space of functions and the space of tempered distributions on $\mathbb{R}^d$, respectively. As usual, we denote by $\mathbb{L}^p$ and $\mathbb{W}^{\sigma,p}$ the Lebesgue and Sobolev spaces $L^p(\mathbb{R}^d;\mathbb{R}^3)$ and $W^{\sigma,p}(\mathbb{R}^d;\mathbb{R}^3)$, with $\mathbb{H}^\sigma := \mathbb{W}^{\sigma,2}$. For the weight function $\rho : \mathbb{R}^d \ni x \mapsto (1+|x|^2)^{-2} \in (0,1]$,

$$|\rho^{-1}\partial_i \rho|_{\mathbb{L}^\infty} + |\rho^{-1}\partial_{ij}\rho|_{\mathbb{L}^\infty} \le c, \quad \forall i,j = 1,\dots,d, \tag{2.1}$$

for some constant c. Then we define weighted spaces:

$$\mathbb{L}^p_\rho := \left\{ f : \mathbb{R}^d \to \mathbb{R}^3 \ : \ \left| f\rho^{\frac{1}{p}} \right|_{\mathbb{L}^p} < \infty \right\}, \quad p \in [1,\infty),$$
$$\mathbb{H}^1_\rho := \left\{ f \in \mathbb{L}^2_\rho \ : \ \nabla f \in \mathbb{L}^2_\rho \right\}.$$

For normed spaces E_1 and E_2, we write $E_1 \hookrightarrow E_2$ if E_1 is continuously embedded in E_2, and $E_1 \Subset E_2$ if E_1 is compactly embedded in E_2. We will use the notation ∇_g for the operator $\nabla_g \phi = g \cdot \nabla \phi$.

Let $h \ge 0$ and let $v \in W^{2,\infty}(\mathbb{R}^d;\mathbb{R}^d)$ be a divergence-free spin velocity field, i.e., $\operatorname{div}(v) = 0$. Let W be an $H^4(\mathbb{R}^d;\mathbb{R}^d)$-valued Wiener process with finite trace-class covariance Q, in the form

$$W(t) = \sum_{k=1}^\infty q_k W_k(t) f_k,$$

where $\{W_k\}$ is a family of real-valued independent Brownian motions, $\{f_k\}$ is a complete orthonormal system in $H^4(\mathbb{R}^d;\mathbb{R}^d)$ consisting of eigenvectors of Q such that $Qf_k = q_k^2 f_k$ for some bounded real q_k and $\operatorname{Tr}(Q) = \sum_{k=1}^\infty q_k^2 < \infty$. For every $k \ge 1$, let $g_k := q_k f_k$. We assume that

$$q^2 := \sum_{k=1}^\infty |g_k|^2_{H^4(\mathbb{R}^d;\mathbb{R}^d)} = \sum_{k=1}^\infty q_k^2 < \infty. \tag{2.2}$$

Recall (1.1), for simplicity we write v instead of $(1+\alpha\beta)v$ and let $\gamma := \frac{\alpha-\beta}{1+\alpha\beta}$. Now we write down the formal stochastic LLG equation. Let $T \in (0,\infty)$. We will prove the existence of a pathwise unique solution $M : [0,T] \times \mathbb{R}^d \to \mathbb{S}^2$ (see Theorem 2.1) to the following equation:

$$\begin{aligned} \mathrm{d}M(t) &= M \times (\Delta M + \Delta^2 M - he_3)\,\mathrm{d}t + \alpha M \times (M \times (\Delta M + \Delta^2 M - he_3))\,\mathrm{d}t \\ &\quad - (\nabla_v M + \gamma M \times \nabla_v m)\,\mathrm{d}t \\ &\quad + \frac{1}{2}\sum_{k=1}^\infty (\nabla_{g_k})^2 M\,\mathrm{d}t - \sum_{k=1}^\infty \nabla_{g_k} M\,\mathrm{d}W_k(t), \end{aligned} \tag{2.3}$$

where $M(0) = M_0 : \mathbb{R}^d \to \mathbb{S}^2$ and $\nabla M_0 \in \mathbb{H}^1$. For simplicity, we define drift and diffusion coefficients

$$\begin{aligned} \bar{F}(u) &:= u \times (\Delta u + \Delta^2 u - he_3) + \alpha u \times (u \times (\Delta u + \Delta^2 u - he_3)) - \gamma u \times \nabla_v u, \\ F(u) &:= \bar{F}(u) - \nabla_v u, \\ S_k(u) &:= (\nabla_{g_k})^2 u, \\ G_k(u) &:= -\nabla_{g_k} u, \end{aligned}$$

for any $u \in \mathbb{H}^2$, where the bi-Laplacian terms are identified via their weak forms. More explicitly,

$$\begin{aligned} \left\langle w \times \Delta^2 u, \varphi \right\rangle_{\mathbb{L}^2} &= \langle \Delta u, \Delta\varphi \times w + \varphi \times \Delta w + 2\nabla\varphi \times \nabla w \rangle_{\mathbb{L}^2}, \\ \left\langle w \times (w \times \Delta^2 u), \varphi \right\rangle_{\mathbb{L}^2} &= \langle w \times \Delta u, \varphi \times \Delta w + \Delta\varphi \times w + 2\nabla\varphi \times \nabla w \rangle_{\mathbb{L}^2} + \langle \Delta u, (\varphi \times w) \times \Delta w \rangle_{\mathbb{L}^2} \\ &\quad + 2\langle \nabla w \times \Delta u, \nabla\varphi \times w + \varphi \times \nabla w \rangle_{\mathbb{L}^2}. \end{aligned} \tag{2.4}$$

for $u, w, \varphi \in \mathbb{H}^2$.

2.2. **Main results.** In this section, we state the main theorem for the existence and uniqueness of solution, and prove the preservation of homotopy type as a corollary.

Definition 2.1. *Given $T \in (0,\infty)$, we say that (2.3) has a martingale solution if there exists a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ defined on which an $H^4(\mathbb{R}^d;\mathbb{R}^d)$-valued Wiener process and a $(\mathcal{F}_t)$-progressively measurable process M, such that*

(i) $|M(t,x)| = 1$, a.e.-(t,x), $\mathbb{P}$-a.s. and $M \in \mathcal{C}([0,T];\mathbb{L}^2_\rho)$ satisfies

$$\mathbb{E}\left[|\nabla M(t)|^2_{L^\infty(0,T;\mathbb{H}^1)} + |M \times \Delta^2 M|^2_{L^2(0,T;\mathbb{L}^2)}\right] < \infty,$$

(ii) for every $t \in [0,T]$, the following equality holds $\mathbb{P}$-a.s. in $\mathbb{L}^2_\rho$

$$M(t) = M_0 + \int_0^t F(M(s))\, \mathrm{d}s + \sum_{k=1}^\infty \int_0^t G_k(M(s)) \circ \mathrm{d}W_k(s), \tag{2.5}$$

where the first Bochner integral and the Stratonovich integral are well-defined in $\mathbb{L}^2$.

Theorem 2.1. *There exists a pathwise unique martingale solution $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}, W, M)$ to (2.3) in the sense of Definition 2.1, such that for $p \in [1,\infty)$,*

$$\mathbb{E}\left[|\nabla M|^{2p}_{L^\infty(0,T;\mathbb{H}^1)} + |M \times \Delta^2 M|^{2p}_{L^2(0,T;\mathbb{L}^2)}\right] < \infty,$$

and $M - e_3 \in \mathcal{C}^\sigma([0,T];\mathbb{L}^2)$ $\mathbb{P}$-a.s. for $\sigma \in (0, \frac{1}{2})$.

The bounds in Theorem 2.1 and a simple interpolation argument implies space-time Hölder continuity almost surely. Hence, $M(t)$ defines a homotopy of maps from $\mathbb{R}^d$. In fact, for $\frac{d}{2} < \tau < 2$,

$$|M(s) - M(t)|^2_{\mathbb{H}^\tau} \le c\,|M(s) - M(t)|^{2-\tau}_{\mathbb{L}^2}\,|M(s) - M(t)|^\tau_{\mathbb{H}^2} \lesssim |s-t|^{\sigma(2-\tau)}$$

and by Sobolev embedding

$$|M(s,x) - M(t,y)| \le c\left(|M(s)|_{\mathbb{H}^\tau}|x-y|^{\tau - \frac{d}{2}} + |M(s) - M(t)|_{\mathbb{H}^\tau}\right)$$

with constants c independent of $0 \le s < t < T$ and $x, y \in \mathbb{R}^d$.

Corollary 2.2. *Trajectories are continuous in space and time and preserve the homotopy type of initial data almost surely.*

3. Approximating equation

Since our desired solution M is not a function in $\mathbb{L}^2$, it is more convenient to define $m := M - e_3$ and to study the following equation for m in $\mathbb{L}^2$, as in [7]:

$$\mathrm{d}m(t) = F(m(t)+e_3)\,\mathrm{d}t + \frac{1}{2}\sum_{k=1}^\infty S_k(m(t))\,\mathrm{d}t + \sum_{k=1}^\infty G_k(m(t))\,\mathrm{d}W_k(t), \quad m(0) = m_0 \in \mathbb{H}^2. \tag{3.1}$$

3.1. **Mollification with cut-off.** For (3.1), we construct an approximating equation using the modified Galerkin method in [24, Chapter 15, Section 7] combined with a cut-off function which controls the vector length of the approximation.

Let J_ε denote a Friedrich mollifier on $\mathbb{R}^d$, such that for $u \in \mathcal{S}(\mathbb{R}^d)$,

$$J_\varepsilon u(x) := (j_\varepsilon * u)(x) = \int_{\mathbb{R}^d} j_\varepsilon(x-y)u(y)\,\mathrm{d}y = \frac{1}{\varepsilon^d}\int_{\mathbb{R}^d} j\left(\frac{x-y}{\varepsilon}\right)u(y)\,\mathrm{d}y,$$

where $j \in \mathcal{S}(\mathbb{R}^d)$ is real-valued and compactly supported with $\int_{\mathbb{R}^d} j(y)\,\mathrm{d}y = 1$.

Fix $R > 1$. Let $\psi_R : [0,\infty) \to [0,1]$ be a smooth non-increasing cut-off function such that $\psi_R(y) = 1$ for $y \in [0,R]$ and $\psi_R(y) = 0$ for $y \ge R+1$. Then for a sufficiently smooth function $u : [0,T]\times\mathbb{R}^d \to \mathbb{R}^3$,

$$\frac{1}{2}\psi_R(|u(t,x)|^2)|u(t,x)|^2 \le R, \quad \forall (t,x) \in [0,T]\times\mathbb{R}^d.$$

Let $f^{(i)}$ denote the i-th derivative of a smooth function $f:\mathbb{R}\to\mathbb{R}$. We have

$$\begin{aligned}\frac{1}{2}\nabla\psi_R(|u|^2) &= \psi_R^{(1)}(|u|^2)\,\langle u,\nabla u\rangle\,,\\ \frac{1}{2}\Delta\psi_R(|u|^2) &= \psi_R^{(1)}(|u|^2)\left(|\nabla u|^2+\langle u,\Delta u\rangle\right)+2\psi_R^{(2)}(|u|^2)\,\langle u,\nabla u\rangle^2\,.\end{aligned}\tag{3.2}$$

Moreover, for $p\geq 0$ and any non-negative integer i, there exists $c(R)=c(R,p)$ such that

$$\psi_R^{(i)}(|u(t,x)|^2)|u(t,x)|^p\leq c(R),\quad \forall (t,x)\in[0,T]\times\mathbb{R}^d,$$

which implies that on $[0,T]\times\mathbb{R}^d$,

$$\begin{aligned}\nabla\psi_R(|u|^2)|u|^p &\leq c(R)|\nabla u|,\\ \Delta\psi_R(|u|^2)|u|^p &\leq c(R)\left(|\nabla u|^2+|\Delta u|\right).\end{aligned}\tag{3.3}$$

For $\varepsilon>0$, consider the approximating equation:

$$\mathrm{d}m_\varepsilon(t)=F_\varepsilon^R(m_\varepsilon(t))\,\mathrm{d}t+\frac{1}{2}\sum_{k=1}^{\infty}S_{k,\varepsilon}(m_\varepsilon(t))\,\mathrm{d}t+\sum_{k=1}^{\infty}G_{k,\varepsilon}(m_\varepsilon(t))\,\mathrm{d}W_k(t),\quad m_\varepsilon(0)=J_\varepsilon m_0,\tag{3.4}$$

where

$$\begin{aligned}F_\varepsilon^R(m_\varepsilon) &= J_\varepsilon\left(\psi_R(|J_\varepsilon m_\varepsilon+e_3|^2)\bar{F}(J_\varepsilon m_\varepsilon+e_3)-\nabla_\nu J_\varepsilon m_\varepsilon\right),\\ S_{k,\varepsilon}(m_\varepsilon) &= J_\varepsilon S_k(J_\varepsilon m_\varepsilon),\\ G_{k,\varepsilon}(m_\varepsilon) &= J_\varepsilon G_k(J_\varepsilon m_\varepsilon).\end{aligned}$$

The cut-off function ψ_R is only required to address the nonlinear term $\bar{F}$. For the linear terms, we integrate-by-parts to obtain desired estimates as shown in the proof of Lemma 3.2.

3.2. **Uniform estimates of m_ε.** We first show that the approximating equation (3.4) admits a unique solution m_ε in Lemma 3.1, and then deduce uniform estimates of m_ε in Lemma 3.2. At the end of this section, we estimate the deviation of $m_\varepsilon+e_3$ from the unit sphere in Lemma 3.3.

Lemma 3.1. *For every $\varepsilon>0$, there exists a unique solution m_ε of (3.4), where m_ε is a progressively measurable process taking values in $\mathbb{H}^2$, such that*

$$\sup_{t\in[0,T]}\mathbb{E}\left[|m_\varepsilon(t)|^2_{\mathbb{H}^2}\right]<\infty,$$

and for every $t\in[0,T]$, the following equality holds $\mathbb{P}$-a.s. in $\mathbb{L}^2$

$$m_\varepsilon(t)=J_\varepsilon m_0+\int_0^t F_\varepsilon^R(m_\varepsilon(s))\,\mathrm{d}s+\frac{1}{2}\sum_{k=1}^{\infty}\int_0^t S_{k,\varepsilon}(m_\varepsilon(s))\,\mathrm{d}s+\sum_{k=1}^{\infty}\int_0^t G_{k,\varepsilon}(m_\varepsilon(s))\,\mathrm{d}W_k(s).$$

Proof. Fix $\varepsilon>0$. We first verify that F_ε^R, $S_{k,\varepsilon}$ and $G_{k,\varepsilon}$ are locally Lipschitz on $\mathbb{H}^2$. Let $u,w\in\mathbb{H}^2$. The derivatives of j are in $\mathcal{S}(\mathbb{R}^2)$, bounded, and in $\mathbb{L}^1$. Thus, for any non-negative integer σ,

$$\begin{aligned}|\nabla^\sigma J_\varepsilon u|_{\mathbb{H}^2} &= |(\nabla^\sigma j_\varepsilon)*(u+\nabla u+\Delta u)|_{\mathbb{L}^2}\\ &\leq |\nabla^\sigma j_\varepsilon|_{\mathbb{L}^1}|u+\nabla u+\Delta u|_{\mathbb{L}^2}\\ &\leq c(\varepsilon)|u|_{\mathbb{H}^2},\end{aligned}$$

and similarly,

$$|\nabla^\sigma J_\varepsilon(u-w)|_{\mathbb{H}^2}\leq c(\varepsilon)|u-w|_{\mathbb{H}^2}.$$

Recall that $\mathbb{H}^2\hookrightarrow\mathbb{L}^\infty$. Let f be a locally Lipschitz function on $\mathbb{H}^2$ with $f(0)=0$, then

$$\begin{aligned}|u\times f(u)-w\times f(w)|_{\mathbb{H}^2} &= |(u-w)\times f(u)|_{\mathbb{H}^2}+|w\times(f(u)-f(w))|_{\mathbb{H}^2}\\ &\lesssim |u-w|_{\mathbb{H}^2}|f(u)|_{\mathbb{H}^2}+|w|_{\mathbb{H}^2}|f(u)-f(w)|_{\mathbb{H}^2}.\end{aligned}$$

Similar arguments follow for scalar products. Then with the Lipschitz property of J_ε and ψ_R, it is clear that F_ε^R, $G_{k,\varepsilon}$ and $S_{k,\varepsilon}$ are locally Lipschitz on $\mathbb{H}^2$ for $k\geq 1$.

Next, let $\mathcal{E}$ denote the space of $\mathbb{H}^2$-valued progressively measurable processes, with the norm

$$|u|_{\mathcal{E}}^2 = \sup_{t\in[0,T]} \mathbb{E}\left[|u(t)|_{\mathbb{H}^2}^2\right].$$

For $n \in \mathbb{N}$, let $F_{\varepsilon,n}^R$, $S_{k,\varepsilon,n}$ and $G_{k,\varepsilon,n}$ denote the Lipschitz modifications of F_ε^R, $S_{k,\varepsilon}$ and $G_{k,\varepsilon}$ on $\mathbb{H}^2$, respectively. For example, we set

$$F_{\varepsilon,n}^R(u) = \begin{cases} F_\varepsilon^R(u) & \text{if } |u|_{\mathbb{H}^2} \le n, \\ F_\varepsilon^R\left(\frac{nu}{|u|_{\mathbb{H}^2}}\right) & \text{if } |u|_{\mathbb{H}^2} > n. \end{cases}$$

Let $A_n : \mathcal{E} \to \mathcal{E}$ be given by

$$\begin{aligned} A_n(u)(t) &= m_0 + \int_0^t \left(F_{\varepsilon,n}^R(u(s)) + \frac{1}{2}\sum_{k=1}^\infty S_{k,\varepsilon,n}(u(s))\right) \mathrm{d}s + \sum_{k=1}^\infty \int_0^t G_{k,\varepsilon,n}(u(s))\, \mathrm{d}W_k(s) \\ &= m_0 + I_n(t) + M_n(t), \end{aligned}$$

where $I_n, M_n \in \mathcal{E}$ by the Lipschitz continuity of $F_{\varepsilon,n}^R$, $S_{k,\varepsilon,n}$ and $G_{k,\varepsilon,n}$. In particular, M_n is an $\mathbb{H}^2$-valued continuous martingale. Once we verify the Lipschitz property of A_n on $\mathcal{E}$, standard arguments using Banach fixed point theorem and localisation by stopping times show that there exists a unique solution m_ε in $\mathcal{E}$ (the limit over n) to the ε-approximating equation (3.4). □

In the following lemma, we deduce an $\mathbb{H}^2$-estimate of m_ε which dominates the energy norm $E(m_\varepsilon + e_3)$ but allows us to directly apply Gronwall's inequality to obtain desired estimates.

Lemma 3.2. *Let $|m_0|_{\mathbb{H}^2} \le c_0$. Then for $p \in [1,\infty)$, there exists a constant $c = c(c_0, p, T)$ independent of ε and R such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |m_\varepsilon(t)|_{\mathbb{H}^2}^{2p}\right] \le c e^{cR^p},$$

and for $\psi_\varepsilon^R := \psi_R(|J_\varepsilon m_\varepsilon + e_3|^2)$,

$$\mathbb{E}\left[\left(\int_0^T |(\psi_\varepsilon^R)^{\frac{1}{2}}(J_\varepsilon m_\varepsilon + e_3) \times \Delta^2 J_\varepsilon m_\varepsilon|_{\mathbb{L}^2}^2(t)\, \mathrm{d}t\right)^p\right] \le c(1+R^p)e^{cR^p}.$$

Proof. Let $u_\varepsilon := J_\varepsilon m_\varepsilon$ and note that $\langle u, J_\varepsilon w\rangle_{\mathbb{L}^2} = \langle J_\varepsilon u, w\rangle_{\mathbb{L}^2}$ for any $u, w \in \mathbb{L}^2$. Let $\psi_\varepsilon^{R,(i)} := \psi_R^{(i)}(|J_\varepsilon m_\varepsilon + e_3|^2)$. We divide the proof into several steps.

Step 1. Estimate $\frac{1}{2}|m_\varepsilon(t)|_{\mathbb{L}^2}^2$.

$$\begin{aligned} \frac{1}{2}\mathrm{d}|m_\varepsilon(t)|_{\mathbb{L}^2}^2 &= \left\langle m_\varepsilon, F_\varepsilon^R(m_\varepsilon)\right\rangle_{\mathbb{L}^2} \mathrm{d}t + \frac{1}{2}\sum_k \left(\left\langle m_\varepsilon, S_{k,\varepsilon}(m_\varepsilon)\right\rangle_{\mathbb{L}^2} + |G_{k,\varepsilon}(m_\varepsilon)|_{\mathbb{L}^2}^2\right) \mathrm{d}t \\ &\quad + \sum_k \left\langle m_\varepsilon, G_{k,\varepsilon}(m_\varepsilon)\right\rangle_{\mathbb{L}^2} \mathrm{d}W_k(t) \\ &=: \mathrm{I}_1\, \mathrm{d}t + \frac{1}{2}\sum_k \mathrm{I}_{2,k}\, \mathrm{d}t + \sum_k \mathrm{I}_{3,k}\, \mathrm{d}W_k(t). \end{aligned}$$

The assumption $\mathrm{div} v = 0$ and (A.3) yield $\langle u_\varepsilon, \nabla_v u_\varepsilon\rangle_{\mathbb{L}^2} = 0$. Then for $\delta \in (0,1)$ and $R > 1$, we have

$$\begin{aligned} \mathrm{I}_1 &= \left\langle m_\varepsilon, F_\varepsilon^R(m_\varepsilon)\right\rangle_{\mathbb{L}^2} \\ &= \left\langle u_\varepsilon, \psi_\varepsilon^R \bar{F}(u_\varepsilon + e_3) - \nabla_v u_\varepsilon\right\rangle_{\mathbb{L}^2} \\ &= \left\langle u_\varepsilon, \psi_\varepsilon^R (u_\varepsilon + e_3) \times (\Delta u_\varepsilon + \alpha(u_\varepsilon + e_3) \times (\Delta u_\varepsilon - h e_3))\right\rangle_{\mathbb{L}^2} \\ &\quad + \left\langle u_\varepsilon, \psi_\varepsilon^R (u_\varepsilon + e_3) \times \left(\Delta^2 u_\varepsilon + \alpha(u_\varepsilon + e_3) \times \Delta^2 u_\varepsilon\right)\right\rangle_{\mathbb{L}^2} \\ &\quad - \gamma\left\langle u_\varepsilon, \psi_\varepsilon^R (u_\varepsilon + e_3) \times \nabla_v u\right\rangle_{\mathbb{L}^2} \\ &\quad - \left\langle u_\varepsilon, \nabla_v u_\varepsilon\right\rangle_{\mathbb{L}^2} \end{aligned}$$

$$\leq \delta |(\psi_\varepsilon^R)^{\frac{1}{2}}(u_\varepsilon + e_3) \times \Delta^2 u_\varepsilon|_{\mathbb{L}^2}^2 + c(\delta) R |u_\varepsilon|_{\mathbb{H}^2}^2.$$

The Stratonovich part has a similar term with opposite sign to the Itô correction. Integrating-by-parts,

$$\begin{aligned}
\langle m_\varepsilon, S_{k,\varepsilon}(m_\varepsilon)\rangle_{\mathbb{L}^2} &= \langle u_\varepsilon, (\nabla_{g_k})^2 u_\varepsilon\rangle_{\mathbb{L}^2} \\
&= -\langle \nabla_{g_k} u_\varepsilon + (\operatorname{div} g_k) u_\varepsilon, \nabla_{g_k} u_\varepsilon\rangle_{\mathbb{L}^2} \\
&\leq -|\nabla_{g_k} u_\varepsilon|_{\mathbb{L}^2}^2 + |g_k|_{W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)}^2 |u_\varepsilon|_{\mathbb{L}^2} |\nabla u_\varepsilon|_{\mathbb{L}^2},
\end{aligned}$$

where $|G_{k,\varepsilon}(m_\varepsilon)|_{\mathbb{L}^2}^2 = |J_\varepsilon \nabla_{g_k} u_\varepsilon|_{\mathbb{L}^2}^2 \leq |\nabla_{g_k} u_\varepsilon|_{\mathbb{L}^2}^2$, leaving

$$\sum_k \mathrm{I}_{2,k} \lesssim \sum_k |g_k|_{W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)}^2 |u_\varepsilon|_{\mathbb{H}^1}^2.$$

Similarly, for the diffusion term, by (A.3),

$$\begin{aligned}
\mathrm{I}_{3,k} = -\langle u_\varepsilon, \nabla_{g_k} u_\varepsilon\rangle_{\mathbb{L}^2} &= \frac{1}{2}\int_{\mathbb{R}^d} (\operatorname{div} g_k)|u_\varepsilon|^2 \, \mathrm{d}x \\
&\leq \frac{1}{2}|g_k|_{W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)} |u_\varepsilon|_{\mathbb{L}^2}^2,
\end{aligned}$$

and by the Burkholder-Davis-Gundy inequality, for $p \in [1,\infty)$,

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t]}\left|2\int_0^s \sum_k \mathrm{I}_{3,k} \, \mathrm{d}W_k(r)\right|^p\right] &\leq c b_p \mathbb{E}\left[\left(\sum_k |g_k|_{W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)}^2 \int_0^t |u_\varepsilon|_{\mathbb{L}^2}^4 \, \mathrm{d}s\right)^{\frac{p}{2}}\right] \\
&\leq c b_p \mathbb{E}\left[\sup_{s\in[0,t]} |u_\varepsilon(s)|_{\mathbb{L}^2}^p \left(\sum_k |g_k|_{W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)}^2 \int_0^t |u_\varepsilon|_{\mathbb{L}^2}^2 \, \mathrm{d}s\right)^{\frac{p}{2}}\right] \\
&\leq \delta^p \mathbb{E}\left[\sup_{s\in[0,t]} |u_\varepsilon(s)|_{\mathbb{L}^2}^{2p}\right] \\
&\quad + c\delta^{-p} b_p^2 \left(\sum_k |g_k|_{W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)}^2\right)^p \mathbb{E}\left[\int_0^t \sup_{r\in[0,s]} |u_\varepsilon(r)|_{\mathbb{L}^2}^{2p} \, \mathrm{d}s\right].
\end{aligned}$$

Step 2. Estimate $\frac{1}{2}|\nabla m_\varepsilon(t)|_{\mathbb{H}^1}^2$.

$$\begin{aligned}
\frac{1}{2}\mathrm{d}\left(|\nabla m_\varepsilon|_{\mathbb{L}^2}^2 + |\Delta m_\varepsilon|_{\mathbb{L}^2}^2\right) &= \langle -\Delta m_\varepsilon + \Delta^2 m_\varepsilon, F_\varepsilon^R(m_\varepsilon)\rangle_{\mathbb{L}^2} \, \mathrm{d}t \\
&\quad + \frac{1}{2}\sum_k \left(\langle -\Delta m_\varepsilon, S_{k,\varepsilon}(m_\varepsilon)\rangle_{\mathbb{L}^2} + |\nabla G_{k,\varepsilon}(m_\varepsilon)|_{\mathbb{L}^2}^2\right) \, \mathrm{d}t \\
&\quad + \frac{1}{2}\sum_k \left(\langle \Delta^2 m_\varepsilon, S_{k,\varepsilon}(m_\varepsilon)\rangle_{\mathbb{L}^2} + |\Delta G_{k,\varepsilon}(m_\varepsilon)|_{\mathbb{L}^2}^2\right) \, \mathrm{d}t \\
&\quad + \sum_k \langle -\Delta m_\varepsilon + \Delta^2 m_\varepsilon, G_{k,\varepsilon}(m_\varepsilon)\rangle_{\mathbb{L}^2} \, \mathrm{d}W_k(t) \\
&=: \mathrm{II}_1 \, \mathrm{d}t + \frac{1}{2}\sum_k \mathrm{II}_{2,k} \, \mathrm{d}t + \sum_k \mathrm{II}_{3,k} \, \mathrm{d}W_k(t).
\end{aligned}$$

We address each term in the following calculations (i) – (iii).

(i) Estimate II_1.

$$\begin{aligned}
\mathrm{II}_1 &= \langle -\Delta m_\varepsilon + \Delta^2 m_\varepsilon, F_\varepsilon^R(m_\varepsilon)\rangle_{\mathbb{L}^2} \\
&= \langle -\Delta u_\varepsilon + \Delta^2 u_\varepsilon, \psi_\varepsilon^R \bar{F}(u_\varepsilon + e_3) - \nabla_\nu u_\varepsilon\rangle_{\mathbb{L}^2} \\
&= \langle -\Delta u_\varepsilon, \psi_\varepsilon^R (u_\varepsilon + e_3) \times \Delta^2 u_\varepsilon\rangle_{\mathbb{L}^2} + \langle \Delta^2 u_\varepsilon, \psi_\varepsilon^R (u_\varepsilon + e_3) \times \Delta u_\varepsilon\rangle_{\mathbb{L}^2} \\
&\quad + \alpha |(\psi_\varepsilon^R)^{\frac{1}{2}}(u_\varepsilon + e_3) \times \Delta u_\varepsilon|_{\mathbb{L}^2}^2 + \alpha \langle -\Delta u_\varepsilon, \psi_\varepsilon^R (u_\varepsilon + e_3) \times \left((u_\varepsilon + e_3) \times \Delta^2 u_\varepsilon\right)\rangle_{\mathbb{L}^2} \\
&\quad - \alpha |(\psi_\varepsilon^R)^{\frac{1}{2}}(u_\varepsilon + e_3) \times \Delta^2 u_\varepsilon|_{\mathbb{L}^2}^2 + \alpha \langle \Delta^2 u_\varepsilon, \psi_\varepsilon^R (u_\varepsilon + e_3) \times ((u_\varepsilon + e_3) \times \Delta u_\varepsilon)\rangle_{\mathbb{L}^2}
\end{aligned}$$

$$
\begin{aligned}
&+\left\langle \Delta u_\varepsilon-\Delta^2 u_\varepsilon, h\psi_\varepsilon^R(u_\varepsilon+e_3)\times(-u_\varepsilon+(u_\varepsilon\times e_3))\right\rangle_{\mathbb{L}^2}\\
&+\gamma\left\langle \Delta u_\varepsilon-\Delta^2 u_\varepsilon, \psi_\varepsilon^R(u_\varepsilon+e_3)\times\nabla_v u_\varepsilon\right\rangle_{\mathbb{L}^2}\\
&+\left\langle \Delta u_\varepsilon-\Delta^2 u_\varepsilon, \nabla_v u_\varepsilon\right\rangle_{\mathbb{L}^2}\\
\le{}& -(\alpha-\delta)|(\psi_\varepsilon^R)^{\frac12}(u_\varepsilon+e_3)\times\Delta^2 u_\varepsilon|^2_{\mathbb{L}^2}+c(\delta)\left(|u_\varepsilon|^2_{\mathbb{L}^2}+|v|_{L^\infty(\mathbb{R}^d;\mathbb{R}^d)}|\nabla u_\varepsilon|^2_{\mathbb{L}^2}+(1+R)|\Delta u_\varepsilon|^2_{\mathbb{L}^2}\right)\\
&+\left\langle \Delta u_\varepsilon-\Delta^2 u_\varepsilon, \nabla_v u_\varepsilon\right\rangle_{\mathbb{L}^2},
\end{aligned}
$$

where

$$
\langle \Delta u_\varepsilon, \nabla_v u_\varepsilon\rangle_{\mathbb{L}^2}\le |v|_{L^\infty(\mathbb{R}^d;\mathbb{R}^d)}|\nabla u_\varepsilon|_{\mathbb{L}^2}|\Delta u_\varepsilon|_{\mathbb{L}^2},
$$

and with $\operatorname{div} v=0$, we have $\langle \Delta u_\varepsilon, \nabla_v\Delta u_\varepsilon\rangle_{\mathbb{L}^2}=0$ and

$$
\begin{aligned}
\left\langle \Delta^2 u_\varepsilon, \nabla_v u_\varepsilon\right\rangle_{\mathbb{L}^2} &= \langle \Delta u_\varepsilon, \Delta\nabla_v u_\varepsilon\rangle_{\mathbb{L}^2}\\
&= \langle \Delta u_\varepsilon, \nabla_v\Delta u_\varepsilon+\nabla_{\Delta v}u_\varepsilon\rangle_{\mathbb{L}^2}+2\sum_{i,j=1}^d\left\langle \Delta u_\varepsilon, \partial_i v_j\partial_{ij}u_\varepsilon\right\rangle_{\mathbb{L}^2}\\
&\lesssim |v|_{W^{2,\infty}(\mathbb{R}^d;\mathbb{R}^d)}|\nabla u_\varepsilon|^2_{\mathbb{H}^1}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mathrm{II}_1\le{}& -(\alpha-\delta)|(\psi_\varepsilon^R)^{\frac12}(u_\varepsilon+e_3)\times\Delta^2 u_\varepsilon|^2_{\mathbb{L}^2}\\
&+c(\delta)(1+|v|_{W^{2,\infty}(\mathbb{R}^d;\mathbb{R}^d)})\left(|u_\varepsilon|^2_{\mathbb{H}^1}+(1+R)|\Delta u_\varepsilon|^2_{\mathbb{L}^2}\right).
\end{aligned}
$$

(ii) Estimate $\mathrm{II}_{2,k}$.

$$
\begin{aligned}
\mathrm{II}_{2,k}&=\left(-\left\langle \Delta m_\varepsilon, S_{k,\varepsilon}(m_\varepsilon)\right\rangle_{\mathbb{L}^2}+|\nabla G_{k,\varepsilon}(m_\varepsilon)|^2_{\mathbb{L}^2}\right)+\left(\left\langle \Delta^2 m_\varepsilon, S_{k,\varepsilon}(m_\varepsilon)\right\rangle_{\mathbb{L}^2}+|\Delta G_{k,\varepsilon}(m_\varepsilon)|^2_{\mathbb{L}^2}\right)\\
&=: \mathrm{II}_{2a,k}+\mathrm{II}_{2b,k}.
\end{aligned}
$$

For the Itô corrections, note that J_ε commutes with ∇ and Δ. We have

$$
\begin{aligned}
\mathrm{II}_{2a,k}&=-\left\langle \Delta u_\varepsilon, (\nabla_{g_k})^2 u_\varepsilon\right\rangle_{\mathbb{L}^2}+|J_\varepsilon\nabla(\nabla_{g_k}u_\varepsilon)|^2_{\mathbb{L}^2}\\
&\lesssim |g_k|^2_{W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)}|\nabla u_\varepsilon|^2_{\mathbb{H}^1}.
\end{aligned}
$$

For $\mathrm{II}_{2b,k}$, we first re-write the Stratonovich term:

$$
\left\langle \Delta^2 m_\varepsilon, S_{k,\varepsilon}(m_\varepsilon)\right\rangle_{\mathbb{L}^2}=\left\langle \Delta^2 u_\varepsilon, (\nabla_{g_k})^2 u_\varepsilon\right\rangle_{\mathbb{L}^2}.
$$

Then we estimate the $\Delta G_{k,\varepsilon}$ part. By Lemma A.1 and (A.13), taking $u=u_\varepsilon$ and $f=g_k$, we obtain

$$
\begin{aligned}
|\Delta G_{k,\varepsilon}(m_\varepsilon)|^2_{\mathbb{L}^2}&=|J_\varepsilon\Delta\nabla_{g_k}u_\varepsilon|^2_{\mathbb{L}^2}\\
&\le |\Delta\nabla_{g_k}u_\varepsilon|^2_{\mathbb{L}^2}\\
&=\left\langle \Delta^2\nabla_{g_k}u_\varepsilon, \nabla_{g_k}u_\varepsilon\right\rangle_{\mathbb{L}^2}\\
&=\left\langle \nabla_{g_k}\Delta^2 u_\varepsilon+4\mathrm{T}_{1a}(u_\varepsilon)+2\mathrm{T}_{1b}(u_\varepsilon)+\mathrm{T}_{1c}(u_\varepsilon), \nabla_{g_k}u_\varepsilon\right\rangle_{\mathbb{L}^2}\\
&=-\left\langle \Delta^2 u_\varepsilon, (\nabla_{g_k})^2 u_\varepsilon\right\rangle_{\mathbb{L}^2}-\left\langle \Delta^2 u_\varepsilon, (\operatorname{div} g_k)\nabla_{g_k}u_\varepsilon\right\rangle_{\mathbb{L}^2}\\
&\quad+\left\langle 4\mathrm{T}_{1a}(u_\varepsilon)+2\mathrm{T}_{1b}(u_\varepsilon)+\mathrm{T}_{1c}(u_\varepsilon), \nabla_{g_k}u_\varepsilon\right\rangle_{\mathbb{L}^2}\\
&\le -\left\langle \Delta^2 m_\varepsilon, S_{k,\varepsilon}(m_\varepsilon)\right\rangle_{\mathbb{L}^2}+c|g_k|^2_{H^4(\mathbb{R}^d;\mathbb{R}^d)}|\nabla u_\varepsilon|^2_{\mathbb{H}^1}.
\end{aligned}
$$

Thus,

$$
\sum_k \mathrm{II}_{2,k}\lesssim\sum_k |g_k|^2_{H^4(\mathbb{R}^d;\mathbb{R}^d)}|\nabla u_\varepsilon|^2_{\mathbb{H}^1}
$$

(iii) Estimate the diffusion part $\mathrm{II}_{3,k}$.

$$
\begin{aligned}
\mathrm{II}_{3,k}&=-\left\langle \Delta m_\varepsilon, G_{k,\varepsilon}(m_\varepsilon)\right\rangle_{\mathbb{L}^2}+\left\langle \Delta^2 m_\varepsilon, G_{k,\varepsilon}(m_\varepsilon)\right\rangle_{\mathbb{L}^2}\\
&=: \mathrm{II}_{3a,k}+\mathrm{II}_{3b,k}.
\end{aligned}
$$

We have

$$\mathrm{II}_{3a,k} = \left\langle \Delta u_\varepsilon, \nabla_{g_k} u_\varepsilon \right\rangle_{\mathbb{L}^2} \le |g_k|_{L^\infty(\mathbb{R}^d;\mathbb{R}^d)} |\Delta u_\varepsilon|_{\mathbb{L}^2} |\nabla u_\varepsilon|_{\mathbb{L}^2},$$

and

$$\begin{aligned}\mathrm{II}_{3b,k} &= -\left\langle \Delta^2 u_\varepsilon, \nabla_{g_k} u_\varepsilon \right\rangle_{\mathbb{L}^2} \\ &= -\left\langle \Delta u_\varepsilon, \Delta \nabla_{g_k} u_\varepsilon \right\rangle_{\mathbb{L}^2} \\ &= -\left\langle \Delta u_\varepsilon, \nabla_{g_k} \Delta u_\varepsilon + \nabla_{\Delta g_k} u_\varepsilon \right\rangle_{\mathbb{L}^2} - 2\sum_{i,j=1}^d \left\langle \Delta u_\varepsilon, \partial_i g_j \partial_{ij} u_\varepsilon \right\rangle_{\mathbb{L}^2} \\ &\le -\left\langle \Delta u_\varepsilon, \nabla_{g_k} \Delta u_\varepsilon \right\rangle_{\mathbb{L}^2} + c|g_k|_{W^{2,\infty}(\mathbb{R}^d;\mathbb{R}^d)} \left(|\Delta u_\varepsilon|_{\mathbb{L}^2} |\nabla u_\varepsilon|_{\mathbb{L}^2} + |\Delta u_\varepsilon|^2_{\mathbb{L}^2} \right),\end{aligned}$$

where

$$\begin{aligned}-\left\langle \Delta u_\varepsilon, \nabla_{g_k} \Delta u_\varepsilon \right\rangle_{\mathbb{L}^2} &= \left\langle \Delta u_\varepsilon, \nabla_{g_k} \Delta u_\varepsilon + (\mathrm{div} g_k) \Delta u_\varepsilon \right\rangle_{\mathbb{L}^2} \\ &= \frac{1}{2} \left\langle \Delta u_\varepsilon, (\mathrm{div} g_k) \Delta u_\varepsilon \right\rangle_{\mathbb{L}^2} \\ &\lesssim |g_k|_{W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)} |\Delta u_\varepsilon|^2_{\mathbb{L}^2}.\end{aligned}$$

Then by Burkholder-Davis-Gundy inequality, for $p \in [1,\infty)$,

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[0,t]} \left| 2\int_0^s \sum_k \mathrm{II}_{3,k}\, \mathrm{d}W_k(r) \right|^p \right] &\le c b_p \mathbb{E}\left[\left(\int_0^t \sum_k |g_k|^2_{W^{2,\infty}(\mathbb{R}^d;\mathbb{R}^d)} |\Delta u_\varepsilon|^2_{\mathbb{L}^2} |\nabla u_\varepsilon|^2_{\mathbb{H}^1}\, \mathrm{d}s \right)^{\frac{p}{2}} \right] \\ &\le \delta^p \mathbb{E}\left[\sup_{s\in[0,t]} |\nabla u_\varepsilon(s)|^{2p}_{\mathbb{H}^1} \right] \\ &\quad + c\delta^{-p} b_p^2 \left(\sum_k |g_k|^2_{W^{2,\infty}(\mathbb{R}^d;\mathbb{R}^d)} \right)^p \mathbb{E}\left[\int_0^t \sup_{r\in[0,s]} |\Delta u_\varepsilon(r)|^{2p}_{\mathbb{L}^2}\, \mathrm{d}s \right].\end{aligned}$$

Step 3. Combine estimates of $\frac{1}{2}|m_\varepsilon(t)|^2_{\mathbb{H}^2}$.

Recall the assumption (2.2). For $R > 1$, we have

$$\begin{aligned}&|m_\varepsilon(t)|^2_{\mathbb{H}^2} + 2(\alpha - 2\delta) |(\psi_\varepsilon^R)^{\frac{1}{2}} (u_\varepsilon + e_3) \times \Delta^2 u_\varepsilon|^2_{\mathbb{L}^2}(t) \\ &\le |J_\varepsilon m_0|^2_{\mathbb{H}^2} + c\int_0^t R|u_\varepsilon(s)|^2_{\mathbb{H}^2}\, \mathrm{d}s + 2\sup_{s\in[0,t]} \left| \int_0^s \sum_k (\mathrm{I}_{3,k} + \mathrm{II}_{3,k})\, \mathrm{d}W_k(r) \right|,\end{aligned}$$

where c depends on $|v|_{W^{2,\infty}(\mathbb{R}^d;\mathbb{R}^d)}, \sum_k |g_k|^2_{H^4(\mathbb{R}^d;\mathbb{R}^d)}, \delta^{-1}, T$, but not on n or R.

Let $\delta \in (0,\frac{1}{4})$ be sufficiently small such that $\alpha - 2\delta > \frac{1}{2}$. Then since $|u_\varepsilon|_{\mathbb{H}^\sigma} \le |m_\varepsilon|_{\mathbb{H}^\sigma}$ for any $\sigma \ge 0$, we have for $p \in [1,\infty)$ that

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]} |m_\varepsilon(t)|^{2p}_{\mathbb{H}^2} + \left(\int_0^T |(\psi_\varepsilon^R)^{\frac{1}{2}} (J_\varepsilon m_\varepsilon + e_3) \times \Delta^2 J_\varepsilon m_\varepsilon|^2_{\mathbb{L}^2}(t)\, \mathrm{d}t \right)^p \right] \\ &\le c\mathbb{E}\left[|J_\varepsilon m_0|^{2p}_{\mathbb{H}^2} + \int_0^T R^p \sup_{s\in[0,t]} |m_\varepsilon(s)|^{2p}_{\mathbb{H}^2}\, \mathrm{d}t \right] + 2^p \delta^p \mathbb{E}\left[\sup_{t\in[0,T]} |m_\varepsilon(t)|^{2p}_{\mathbb{H}^2} \right],\end{aligned}$$

where $1 - 2^p\delta^p > \frac{1}{2}$, and then the last expectation term on the right-hand side can be absorbed into the left-hand side. By Gronwall's inequality,

$$\mathbb{E}\left[\sup_{t\in[0,T]} |m_\varepsilon(t)|^{2p}_{\mathbb{H}^2} \right] \le c\mathbb{E}\left[|m_0|^{2p}_{\mathbb{H}^2} \right] e^{cR^p} \le c e^{cR^p},$$

for some constant c independent of n and R. As a result,

$$\mathbb{E}\left[\left(\int_0^T |(\psi_\varepsilon^R)^{\frac{1}{2}}(J_\varepsilon m_\varepsilon + e_3)\times \Delta^2 J_\varepsilon m_\varepsilon|^2_{\mathbb{L}^2}(t)\, \mathrm{d}t\right)^p\right] \le c(1+R^p)e^{cR^p},$$

concluding the proof. □

As a result of Lemma 3.2, $F_\varepsilon^R(m_\varepsilon) + \frac{1}{2}\sum_k S_{k,\varepsilon}(m_\varepsilon) \in L^p(\Omega; L^2(0,T;\mathbb{L}^2))$, and

$$\mathbb{E}\left[\left|\int_0^T \sum_k G_{k,\varepsilon}(m_\varepsilon(t))\, \mathrm{d}W_k(t)\right|^p_{W^{\sigma,p}(0,T;\mathbb{L}^2)}\right] \lesssim \mathbb{E}\left[\left(\int_0^T \sum_k |g_k|^2_{L^\infty(\mathbb{R}^d;\mathbb{R}^d)} |\nabla m_\varepsilon(t)|^2_{\mathbb{L}^2}\, \mathrm{d}t\right)^{\frac{p}{2}}\right] \le c(R),$$

for fixed $R>1$, $\sigma\in(0,\frac{1}{2})$, $p\in[2,\infty)$. Thus, with the embedding $W^{1,2}(0,T;\mathbb{L}^2)\hookrightarrow W^{\sigma,p}(0,T;\mathbb{L}^2)$ for $\sigma - \frac{1}{p} < \frac{1}{2}$, we have

$$\mathbb{E}\left[|m_\varepsilon|^p_{W^{\sigma,p}(0,T;\mathbb{L}^2)}\right] \le c(R). \tag{3.5}$$

Next we examine the vector length of $M_\varepsilon := m_\varepsilon + e_3$. In Lemma 3.3, we show that although M_ε does not take values in $\mathbb{S}^2$, its deviation from $\mathbb{S}^2$ can be controlled by the strength of mollifications. The proof is similar to that in [19] which starts from applying Itô's formula to

$$\varphi(m_\varepsilon) := \frac{1}{4}|1-|m_\varepsilon+e_3|^2|^2_{\mathbb{L}^2_\rho} = \frac{1}{4}|1-|M_\varepsilon|^2|^2_{\mathbb{L}^2_\rho}. \tag{3.6}$$

Here we work with mollified functions and weighted spaces. Thus, we first identify some remainder terms which arise from the difference between the convolution of products and the product of convolutions, before stating and proving Lemma 3.3. Again, let $u_\varepsilon := J_\varepsilon m_\varepsilon$, and $d_\varepsilon(f) := J_\varepsilon f - f$ for any $f:[0,T]\times\mathbb{R}^d \to \mathbb{R}$ or $\mathbb{R}^3$. Then

$$\begin{aligned} p_\varepsilon(m_\varepsilon) &:= J_\varepsilon\left((1-|m_\varepsilon+e_3|^2)(m_\varepsilon+e_3)\rho\right) - (1-|J_\varepsilon m_\varepsilon + e_3|^2)(J_\varepsilon m_\varepsilon+e_3)\rho \\ &= J_\varepsilon\left((1-|M_\varepsilon|^2)M_\varepsilon\rho\right) - (1-|u_\varepsilon+e_3|^2)(u_\varepsilon+e_3)\rho \\ &= d_\varepsilon((1-|M_\varepsilon|^2)M_\varepsilon\rho) - (1-|u_\varepsilon+e_3|^2)d_\varepsilon(m_\varepsilon)\rho + \langle d_\varepsilon(m_\varepsilon), m_\varepsilon+u_\varepsilon+2e_3\rangle M_\varepsilon\rho. \end{aligned} \tag{3.7}$$

Since $|d_\varepsilon(f(t))|_{\mathbb{L}^2} \lesssim |f(t)|_{\mathbb{L}^2}$ for any $t\in[0,T]$, using Lemma 3.2 and the continuous embeddings $\mathbb{H}^2\hookrightarrow\mathbb{L}^\infty$ and $\mathbb{H}^1\hookrightarrow\mathbb{L}^4$, we have that for any $t\in[0,T]$ and $p\in[1,\infty)$,

$$\begin{aligned} &\mathbb{E}\left[\int_0^t |p_\varepsilon(m_\varepsilon)|^{2p}_{\mathbb{L}^2}\, \mathrm{d}s\right] \\ &\lesssim \mathbb{E}\left[\int_0^t \left(|M_\varepsilon|_{\mathbb{L}^2_\rho} + |M_\varepsilon|_{\mathbb{L}^\infty}|M_\varepsilon|^2_{\mathbb{L}^4_\rho} + (1+|u_\varepsilon|_{\mathbb{L}^\infty})|m_\varepsilon|_{\mathbb{L}^2} + |M_\varepsilon|_{\mathbb{L}^\infty}\left(|m_\varepsilon|_{\mathbb{L}^\infty}|m_\varepsilon+u_\varepsilon|_{\mathbb{L}^2} + |m_\varepsilon|_{\mathbb{L}^2}\right)\right)^{2p} \mathrm{d}s\right] \\ &\lesssim \mathbb{E}\left[\int_0^t \left((1+|m_\varepsilon|_{\mathbb{H}^2})|m_\varepsilon+e_3|^2_{\mathbb{H}^1_\rho} + (1+|m_\varepsilon|_{\mathbb{H}^2})|m_\varepsilon|_{\mathbb{L}^2} + (1+|m_\varepsilon|_{\mathbb{H}^2})\left(|m_\varepsilon|^2_{\mathbb{H}^2} + |m_\varepsilon|_{\mathbb{L}^2}\right)\right)^{2p} \mathrm{d}s\right] \\ &\le c(R). \end{aligned}$$

In addition,

$$|d_\varepsilon(f)|^4_{\mathbb{L}^4_\rho} \lesssim |f|^2_{\mathbb{L}^\infty}|d_\varepsilon(f)|^2_{\mathbb{L}^2_\rho} \lesssim |f|^2_{\mathbb{L}^\infty}|d_\varepsilon(f)|_{\mathbb{L}^2}|d_\varepsilon(f)|_{\mathbb{L}^2_\rho} \lesssim |f|^3_{\mathbb{H}^2}|d_\varepsilon(f)|_{\mathbb{L}^2_\rho}. \tag{3.8}$$

For remainders of higher-order terms,

$$\begin{aligned} (1-|u_\varepsilon+e_3|^2)^2 &= (1-|M_\varepsilon+d_\varepsilon(m_\varepsilon)|^2)^2 \\ &= (1-|M_\varepsilon|^2)^2 + \left(|d_\varepsilon(m_\varepsilon)|^2 + 2\langle d_\varepsilon(m_\varepsilon), M_\varepsilon\rangle\right)^2 \\ &\quad + 2(|m_\varepsilon|^2 + 2\langle m_\varepsilon, e_3\rangle)\left(|d_\varepsilon(m_\varepsilon)|^2 + 2\langle d_\varepsilon(m_\varepsilon), M_\varepsilon\rangle\right), \end{aligned}$$

Thus,

$$
\begin{aligned}
\left|1-|u_\varepsilon+e_3|^2\right|^2_{\mathbb{L}^2_\rho} &\lesssim \left|1-|M_\varepsilon|^2\right|^2_{\mathbb{L}^2_\rho} + |d_\varepsilon(m_\varepsilon)|^4_{\mathbb{L}^4_\rho} \\
&\quad + \left(|M_\varepsilon|^2_{\mathbb{L}^\infty} + |m_\varepsilon|^2_{\mathbb{L}^\infty} + |m_\varepsilon|_{\mathbb{L}^\infty}\right) |d_\varepsilon(m_\varepsilon)|^2_{\mathbb{L}^2_\rho} \\
&\quad + (|m_\varepsilon|^2_{\mathbb{L}^4} + |m_\varepsilon|_{\mathbb{L}^2})|M_\varepsilon|_{\mathbb{L}^\infty}|d_\varepsilon(m_\varepsilon)|_{\mathbb{L}^2_\rho} \\
&\lesssim \left|1-|M_\varepsilon|^2\right|^2_{\mathbb{L}^2_\rho} + (1+|m_\varepsilon|^3_{\mathbb{H}^2})|d_\varepsilon(m_\varepsilon)|_{\mathbb{L}^2_\rho},
\end{aligned} \tag{3.9}
$$

where the last inequality holds by (3.8) and (A.5).

Lemma 3.3. *There exist constants $c_1 = c_1(R,T) > 0$ and $c_2 = c_2(T) > 0$, both independent of ε, such that*

$$
\mathbb{E}\left[|1-|m_\varepsilon(t)+e_3|^2|^2_{\mathbb{L}^2_\rho}\right] \le c_1\, e^{c_2} \left(|1-|J_\varepsilon m_0+e_3|^2|^2_{\mathbb{L}^2_\rho} + \mathbb{E}\left[\int_0^t A_\varepsilon(m_\varepsilon(s))\, \mathrm{d}s\right]^{\frac{1}{2}} \right),
$$

for any $t \in [0,T]$, where

$$
A_\varepsilon(m_\varepsilon) := |d_\varepsilon(m_\varepsilon)|^2_{\mathbb{H}^1_\rho} + \sum_k |d_\varepsilon(\nabla_{g_k} J_\varepsilon m_\varepsilon)|^2_{\mathbb{L}^2_\rho} + |p_\varepsilon(m_\varepsilon)|^2_{\mathbb{L}^2}. \tag{3.10}
$$

Proof. Recall $\varphi(m_\varepsilon)$ from (3.6). We have $\varphi(m_\varepsilon) \le c(|m_\varepsilon|^4_{\mathbb{L}^4} + |m_\varepsilon|^2_{\mathbb{L}^2})$ and

$$
\begin{aligned}
\varphi'(m_\varepsilon)u &= -\left\langle (1-|M_\varepsilon|^2)M_\varepsilon, u\right\rangle_{\mathbb{L}^2_\rho}, \\
\varphi''(m_\varepsilon)(u,w) &= -\left\langle (1-|M_\varepsilon|^2)u, w\right\rangle_{\mathbb{L}^2_\rho} + 2\int_{\mathbb{R}^d} \langle M_\varepsilon, u\rangle \langle M_\varepsilon, w\rangle\, \rho(x)\, \mathrm{d}x.
\end{aligned}
$$

By Itô's formula,

$$
\begin{aligned}
\frac{1}{4}\mathrm{d}|1-|M_\varepsilon(t)|^2|^2_{\mathbb{L}^2_\rho} &= -\left\langle (1-|M_\varepsilon|^2)M_\varepsilon, F^R_\varepsilon(M_\varepsilon)\right\rangle_{\mathbb{L}^2_\rho}\, \mathrm{d}t \\
&\quad - \frac{1}{2}\sum_k \left\langle (1-|M_\varepsilon|^2)M_\varepsilon, S_{k,\varepsilon}(m_\varepsilon)\right\rangle_{\mathbb{L}^2_\rho}\, \mathrm{d}t \\
&\quad - \frac{1}{2}\sum_k \left\langle (1-|M_\varepsilon|^2)G_{k,\varepsilon}(m_\varepsilon), G_{k,\varepsilon}(m_\varepsilon)\right\rangle_{\mathbb{L}^2_\rho}\, \mathrm{d}t \\
&\quad + \sum_k \int_{\mathbb{R}^d} \left\langle M_\varepsilon, G_{k,\varepsilon}(m_\varepsilon)\right\rangle^2 \rho(x)\, \mathrm{d}x\, \mathrm{d}t \\
&\quad - \sum_k \left\langle (1-|M_\varepsilon|^2)M_\varepsilon, G_{k,\varepsilon}(m_\varepsilon)\right\rangle_{\mathbb{L}^2_\rho}\, \mathrm{d}W_k \\
&=: \mathrm{U}_1\, \mathrm{d}t - \frac{1}{2}\sum_k \left(\mathrm{U}_{2,k} + \mathrm{U}_{3,k} - 2\mathrm{U}_{4,k}\right)\, \mathrm{d}t - \sum_k \mathrm{U}_{5,k}\, \mathrm{d}W_k.
\end{aligned} \tag{3.11}
$$

For the diffusion term $\mathrm{U}_{5,k}$,

$$
\begin{aligned}
\mathrm{U}_{5,k} &= -\left\langle (1-|M_\varepsilon|^2)M_\varepsilon, G_{k,\varepsilon}(m_\varepsilon)\right\rangle_{\mathbb{L}^2_\rho} \\
&\lesssim |1-|M_\varepsilon|^2|_{\mathbb{L}^2_\rho} |M_\varepsilon|_{\mathbb{L}^\infty} |\nabla m_\varepsilon|_{\mathbb{L}^2_\rho} \\
&\lesssim \left(|m_\varepsilon|^4_{\mathbb{L}^4} + |m_\varepsilon|^2_{\mathbb{L}^2}\right)(1+|m_\varepsilon|_{\mathbb{H}^2})\, |\nabla m_\varepsilon|_{\mathbb{L}^2}.
\end{aligned}
$$

Since $m_\varepsilon \in L^{2p}(\Omega; L^\infty(0,T;\mathbb{H}^2))$ for $p \in [1,\infty)$, $\mathrm{U}_{5,k} \in L^{2p}(\Omega; L^2(0,T))$. Then the Itô integral is well-defined, thus $\sum_k \mathrm{U}_{5,k}\, \mathrm{d}W_k$ has zero expectation.

For U_1, we have

$$
\begin{aligned}
\mathrm{U}_1 &= -\left\langle J_\varepsilon\left((1-|M_\varepsilon|^2)M_\varepsilon\rho\right), \psi^R_\varepsilon \bar{F}(u_\varepsilon+e_3) - \nabla_v u_\varepsilon\right\rangle_{\mathbb{L}^2} \\
&= -\left\langle (1-|u_\varepsilon+e_3|^2)(u_\varepsilon+e_3), \psi^R_\varepsilon \bar{F}(u_\varepsilon+e_3)\right\rangle_{\mathbb{L}^2_\rho} \\
&\quad + \left\langle (1-|u_\varepsilon+e_3|^2)(u_\varepsilon+e_3), \nabla_v(u_\varepsilon+e_3)\right\rangle_{\mathbb{L}^2_\rho}
\end{aligned}
$$

$$
\begin{aligned}
&- \left\langle p_\varepsilon(m_\varepsilon), \psi_\varepsilon^R \bar{F}(u_\varepsilon+e_3) - \nabla_v u_\varepsilon \right\rangle_{\mathbb{L}^2} \\
&=: \mathrm{U}_{1a} + \mathrm{U}_{1b} + \mathrm{U}_{1c},
\end{aligned}
$$

where U_{1a} is equal to 0 by the cross-product structure of $\bar{F}$. For U_{1b}, since $\operatorname{div}(v)=0$ and $v\rho$ vanishes at infinity, we have $\operatorname{div}(v\rho)=\langle v,\nabla\rho\rangle$ and $\int_{\mathbb{R}^d}\operatorname{div}(v\rho)\,\mathrm{d}x=0$. Then by (A.4),

$$
\begin{aligned}
\mathrm{U}_{1b} &= \frac{1}{4}\int_{\mathbb{R}^d} \operatorname{div}(v\rho)(1-|u_\varepsilon+e_3|^2)^2\,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^d}\operatorname{div}(v\rho)\,\mathrm{d}x \\
&= \frac{1}{4}\int_{\mathbb{R}^d} \left\langle v, \rho^{-1}\nabla\rho \right\rangle (1-|u_\varepsilon+e_3|^2)^2 \rho\,\mathrm{d}x \\
&\lesssim |v|_{\mathbb{L}^\infty} |\rho^{-1}\nabla\rho|_{\mathbb{L}^\infty} \left|1-|u_\varepsilon+e_3|^2\right|_{\mathbb{L}^2_\rho}^2,
\end{aligned}
$$

where $|\rho^{-1}\nabla\rho|_{\mathbb{L}^\infty}=\frac{1}{2}$. Thus, by (3.9) and the estimates in Lemma 3.2, for any $t\in[0,T]$,

$$
\begin{aligned}
\mathbb{E}\left[\int_0^t \mathrm{U}_1\,\mathrm{d}s\right] &\lesssim \mathbb{E}\left[\int_0^t \left|1-|M_\varepsilon|^2\right|_{\mathbb{L}^2_\rho}^2\,\mathrm{d}s + \int_0^t (1+|m_\varepsilon|_{\mathbb{H}^2}^3)|d_\varepsilon(m_\varepsilon)|_{\mathbb{L}^2_\rho}\,\mathrm{d}s\right] \\
&\quad + \mathbb{E}\left[\int_0^t |p_\varepsilon(m_\varepsilon)|_{\mathbb{L}^2} |\psi_\varepsilon^R \bar{F}(u_\varepsilon+e_3) - \nabla_v u_\varepsilon|_{\mathbb{L}^2}\,\mathrm{d}s\right] \\
&\le c\mathbb{E}\left[\int_0^t \left|1-|M_\varepsilon|^2\right|_{\mathbb{L}^2_\rho}^2\,\mathrm{d}s\right] + c(R)\,\mathbb{E}\left[\int_0^t \left(|d_\varepsilon(m_\varepsilon)|_{\mathbb{L}^2_\rho}^2 + |p_\varepsilon(m_\varepsilon)|_{\mathbb{L}^2}^2\right)\mathrm{d}s\right]^{\frac{1}{2}}.
\end{aligned}
$$

For $\mathrm{U}_{2,k}$,

$$
\begin{aligned}
\mathrm{U}_{2,k} &= \left\langle J_\varepsilon\left((1-|M_\varepsilon|^2)M_\varepsilon\rho\right), S_k(u_\varepsilon)\right\rangle_{\mathbb{L}^2} \\
&= \left\langle (1-|u_\varepsilon+e_3|^2)(u_\varepsilon+e_3), (\nabla_{g_k})^2 u_\varepsilon \right\rangle_{\mathbb{L}^2_\rho} + \left\langle p_\varepsilon(m_\varepsilon), (\nabla_{g_k})^2 u_\varepsilon\right\rangle_{\mathbb{L}^2} \\
&=: \mathrm{U}_{2a,k} + \mathrm{U}_{2b,k},
\end{aligned}
$$

where by (2.2) and the estimates in Lemma 3.2,

$$
\begin{aligned}
\mathbb{E}\left[\sum_k \int_0^t |\mathrm{U}_{2b,k}|\,\mathrm{d}s\right] &\le \mathbb{E}\left[\sum_k |g_k|_{W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)}^2 \int_0^t |p_\varepsilon(m_\varepsilon)|_{\mathbb{L}^2} |\nabla u_\varepsilon|_{\mathbb{H}^1}\,\mathrm{d}s\right] \\
&\le c(R)\,\mathbb{E}\left[\int_0^t |p_\varepsilon(m_\varepsilon)|_{\mathbb{L}^2}^2\,\mathrm{d}s\right]^{\frac{1}{2}}.
\end{aligned}
$$

For $\mathrm{U}_{2a,k}$, integrating-by-parts,

$$
\begin{aligned}
\mathrm{U}_{2a,k} &= \left\langle (1-|u_\varepsilon+e_3|^2)(u_\varepsilon+e_3), (\nabla_{g_k})^2 u_\varepsilon\right\rangle_{\mathbb{L}^2_\rho} \\
&= \left\langle (1-|u_\varepsilon+e_3|^2)(u_\varepsilon+e_3), \nabla_{\rho g_k}\nabla_{g_k} u_\varepsilon\right\rangle_{\mathbb{L}^2} \\
&= -\int_{\mathbb{R}^d} (1-|u_\varepsilon+e_3|^2)|\nabla_{g_k}u_\varepsilon|^2\rho\,\mathrm{d}x + 2\int_{\mathbb{R}^d}\left\langle u_\varepsilon+e_3, \nabla_{g_k}u_\varepsilon\right\rangle^2\rho\,\mathrm{d}x \\
&\quad - \left\langle \operatorname{div}(g_k\rho)(1-|u_\varepsilon+e_3|^2)(u_\varepsilon+e_3), \nabla_{g_k}(u_\varepsilon+e_3)\right\rangle_{\mathbb{L}^2} \\
&=: -\widehat{\mathrm{U}}_{3,k} + 2\widehat{\mathrm{U}}_{4,k} + \mathrm{V}_{2a,k},
\end{aligned}
$$

where $-\widehat{\mathrm{U}}_{3,k}+2\widehat{\mathrm{U}}_{4,k}$ is in a similar form to $-\mathrm{U}_{3,k}+2\mathrm{U}_{4,k}$. For the remainder $\mathrm{V}_{2a,k}$, by (A.4),

$$
\mathrm{V}_{2a,k} = -\frac{1}{4}\left(\int_{\mathbb{R}^d}\operatorname{div}\left(\operatorname{div}(g_k\rho)g_k\right)\rho^{-1}\left|1-|u_\varepsilon+e_3|^2\right|^2\rho\,\mathrm{d}x - \int_{\mathbb{R}^d}\operatorname{div}\left(\operatorname{div}(g_k\rho)g_k\right)\,\mathrm{d}x\right),
$$

where the fact ρ and $\nabla\rho$ vanish at infinity implies $\int_{\mathbb{R}^d}\operatorname{div}(\operatorname{div}(g_k\rho)g_k)\,\mathrm{d}x=0$, and by (2.1),

$$
\begin{aligned}
\operatorname{div}\left(\operatorname{div}(g_k\rho)g_k\right)\rho^{-1} &= (\operatorname{div}g_k)^2 + \left\langle \nabla_{g_k}g_k + 2g_k\operatorname{div}g_k, \rho^{-1}\nabla\rho\right\rangle + \nabla_{g_k}\operatorname{div}g_k + \left\langle g_k, \rho^{-1}\nabla_{g_k}\nabla\rho\right\rangle \\
&\lesssim |g_k|_{W^{2,\infty}(\mathbb{R}^d;\mathbb{R}^d)}^2.
\end{aligned}
$$

Then by (2.2) and (3.9), we are left with

$$\mathbb{E}\left[\int_0^t \sum_k |\mathrm{V}_{2a,k}| \, \mathrm{d}s\right] \le c\mathbb{E}\left[\int_0^t |1-|M_\varepsilon(s)|^2|^2_{\mathbb{L}^2_\rho} \, \mathrm{d}s\right] + c(R)\mathbb{E}\left[\int_0^t |d_\varepsilon(m_\varepsilon)|^2_{\mathbb{L}^2_\rho} \, \mathrm{d}s\right]^{\frac{1}{2}}.$$

Now we estimate $\mathrm{U}_{3,k}$ and $\mathrm{U}_{4,k}$. Note that $G_{k,\varepsilon}(m_\varepsilon) - \nabla_{g_k} u_\varepsilon = d_\varepsilon(\nabla_{g_k} u_\varepsilon)$. We have

$$\begin{aligned}
\mathrm{U}_{3,k} &= \left\langle (1-|M_\varepsilon|^2) G_{k,\varepsilon}(m_\varepsilon), G_{k,\varepsilon}(m_\varepsilon) \right\rangle_{\mathbb{L}^2_\rho} \\
&= \left\langle (1-|u_\varepsilon+e_3|^2) \nabla_{g_k} u_\varepsilon, \nabla_{g_k} u_\varepsilon \right\rangle_{\mathbb{L}^2_\rho} \\
&\quad + \left\langle \langle d_\varepsilon(m_\varepsilon), m_\varepsilon + u_\varepsilon + 2e_3 \rangle \, G_{k,\varepsilon}(m_\varepsilon), G_{k,\varepsilon}(m_\varepsilon) \right\rangle_{\mathbb{L}^2_\rho} \\
&\quad + \left\langle (1-|u_\varepsilon+e_3|^2) d_\varepsilon(\nabla_{g_k} u_\varepsilon), G_{k,\varepsilon}(m_\varepsilon) + \nabla_{g_k} u_\varepsilon \right\rangle_{\mathbb{L}^2_\rho} \\
&= \widehat{\mathrm{U}}_{3,k} + \mathrm{V}_{3,k},
\end{aligned}$$

and thus

$$\begin{aligned}
\mathbb{E}\left[\sum_k \int_0^t |\mathrm{V}_{3,k}| \, \mathrm{d}s\right] &\lesssim \mathbb{E}\left[\sum_k |g_k|^2_{L^\infty(\mathbb{R}^d;\mathbb{R}^d)} \int_0^t |d_\varepsilon(m_\varepsilon)|_{\mathbb{L}^2_\rho} \left(|m_\varepsilon|_{\mathbb{L}^\infty} + 1\right) |\nabla m_\varepsilon|^2_{\mathbb{L}^4} \, \mathrm{d}s\right] \\
&\quad + \mathbb{E}\left[\sum_k |g_k|_{L^\infty(\mathbb{R}^d;\mathbb{R}^d)} \int_0^t |d_\varepsilon(\nabla_{g_k} u_\varepsilon)|_{\mathbb{L}^2_\rho} \left(|m_\varepsilon|^2_{\mathbb{L}^\infty} + |m_\varepsilon|_{\mathbb{L}^\infty}\right) |\nabla m_\varepsilon|_{\mathbb{L}^2} \, \mathrm{d}s\right] \\
&\le c(R)\mathbb{E}\left[\int_0^t \left(|d_\varepsilon(m_\varepsilon)|^2_{\mathbb{L}^2_\rho} + \sum_k |d_\varepsilon(\nabla_{g_k} u_\varepsilon)|^2_{\mathbb{L}^2_\rho}\right) \mathrm{d}s\right]^{\frac{1}{2}}.
\end{aligned}$$

Similarly,

$$\left\langle M_\varepsilon, G_{k,\varepsilon}(m_\varepsilon) \right\rangle - \left\langle u_\varepsilon + e_3, \nabla_{g_k} u_\varepsilon \right\rangle = \left\langle u_\varepsilon + e_3, d_\varepsilon(\nabla_{g_k} u_\varepsilon) \right\rangle - \left\langle d_\varepsilon(m_\varepsilon), G_{k,\varepsilon}(m_\varepsilon) \right\rangle =: r_{4,k}.$$

By (3.8) and (A.5),

$$\begin{aligned}
|r_{4,k}|^2_{\mathbb{L}^2_\rho} &\lesssim (1+|m_\varepsilon|_{\mathbb{L}^\infty})^2 |d_\varepsilon(\nabla_{g_k} u_\varepsilon)|^2_{\mathbb{L}^2_\rho} + |g_k|^2_{L^\infty(\mathbb{R}^d;\mathbb{R}^d)} |\nabla m_\varepsilon|^2_{\mathbb{L}^4} |d_\varepsilon(m_\varepsilon)\rho^{\frac{1}{4}}|^2_{\mathbb{L}^4_\rho} \\
&\lesssim \left(|g_k|_{L^\infty(\mathbb{R}^d;\mathbb{R}^d)}(1+|m_\varepsilon|^3_{\mathbb{H}^2}) + |g_k|^2_{L^\infty(\mathbb{R}^d;\mathbb{R}^d)} |m_\varepsilon|^3_{\mathbb{H}^2}\right) \left(|d_\varepsilon(m_\varepsilon)|_{\mathbb{L}^2_\rho} + |d_\varepsilon(\nabla_{g_k} u_\varepsilon)|_{\mathbb{L}^2_\rho}\right),
\end{aligned}$$

Then for $\mathrm{U}_{4,k}$, we have

$$\begin{aligned}
\mathrm{U}_{4,k} &= \int_{\mathbb{R}^d} \left\langle M_\varepsilon, G_{k,\varepsilon}(m_\varepsilon) \right\rangle^2 \rho \, \mathrm{d}x \\
&= \int_{\mathbb{R}^d} \left\langle u_\varepsilon + e_3, \nabla_{g_k} u_\varepsilon \right\rangle^2 \rho \, \mathrm{d}x + \int_{\mathbb{R}^d} \left(r^2_{4,k} + 2\left\langle u_\varepsilon + e_3, \nabla_{g_k} u_\varepsilon \right\rangle r_{4,k}\right) \rho \, \mathrm{d}x \\
&= \widehat{\mathrm{U}}_{4,k} + \mathrm{V}_{4,k},
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{E}\left[\sum_k \int_0^t |\mathrm{V}_{4,k}| \, \mathrm{d}s\right] &\lesssim \mathbb{E}\left[\sum_k \int_0^t |r_{4,k}|^2_{\mathbb{L}^2_\rho} + 2|g_k|_{\mathbb{L}^\infty(\mathbb{R}^d;\mathbb{R}^d)} (1+|m_\varepsilon|_{\mathbb{L}^\infty}) |\nabla m_\varepsilon|_{\mathbb{L}^2} |r_{4,k}|_{\mathbb{L}^2_\rho} \, \mathrm{d}s\right] \\
&\le c(R)\mathbb{E}\left[\int_0^t \left(|d_\varepsilon(m_\varepsilon)|^2_{\mathbb{L}^2_\rho} + |d_\varepsilon(m_\varepsilon)\rho^{\frac{1}{4}}|^2_{\mathbb{L}^4_\rho} + \sum_k |d_\varepsilon(\nabla_{g_k} u_\varepsilon)|^2_{\mathbb{L}^2_\rho}\right) \mathrm{d}s\right]^{\frac{1}{2}}.
\end{aligned}$$

Note that

$$|d_\varepsilon(m_\varepsilon)\rho^{\frac{1}{4}}|^2_{\mathbb{L}^4_\rho} = |d_\varepsilon(m_\varepsilon)\rho^{\frac{1}{2}}|^2_{\mathbb{L}^4} \lesssim |d_\varepsilon(m_\varepsilon)\rho^{\frac{1}{2}}|^2_{\mathbb{H}^1} = |d_\varepsilon(m_\varepsilon)|^2_{\mathbb{H}^1_\rho}.$$

Hence, the main components of the Stratonovich and Itô corrections cancel with each other, leaving

$$-\mathbb{E}\left[\sum_k \left(\mathrm{U}_{2,k} + \mathrm{U}_{3,k} - 2\mathrm{U}_{4,k}\right) \, \mathrm{d}t\right] \le c\mathbb{E}\left[\int_0^t |1-|M_\varepsilon(s)|^2|^2_{\mathbb{L}^2_\rho} \, \mathrm{d}s\right]$$

$$+c(R)\mathbb{E}\left[\int_0^t \left(|d_\varepsilon(m_\varepsilon)|^2_{\mathbb{H}^1_\rho} + \sum_k |d_\varepsilon(\nabla_{g_k} u_\varepsilon)|^2_{\mathbb{L}^2_\rho} + |p_\varepsilon(m_\varepsilon)|^2_{\mathbb{L}^2}\right) \mathrm{d}s\right]^{\frac{1}{2}},$$

Overall, we obtain

$$\mathbb{E}\left[|1-|M_\varepsilon(t)|^2|^2_{\mathbb{L}^2_\rho}\right] \le |1-|J_\varepsilon m_0 + e_3|^2|^2_{\mathbb{L}^2_\rho} + c\mathbb{E}\left[\int_0^t |1-|M_\varepsilon(s)|^2|^2_{\mathbb{L}^2_\rho}\ \mathrm{d}s\right]$$
$$+c(R)\mathbb{E}\left[\int_0^t A_\varepsilon(m_\varepsilon)\ \mathrm{d}s\right]^{\frac{1}{2}},$$

for any $t \in [0,T]$. Then we apply Gronwall's lemma to yield the desired result. □

4. Convergences for fixed R

In this section, we fix $R > 1$, and employ Skorohod theorem and compactness argument to deduce convergences of (functions of) new approximations $\tilde{m}_\varepsilon$. These convergences and subsequent regularity properties of the limit $\tilde{m}$ allow us to show that $\tilde{m}+e_3$ is a martingale solution of (2.3) later in Section 5.

For Banach spaces B_0, B_1, B_2 and B_3 such that B_0, B_2 are reflexive and $B_0 \Subset B_1 \hookrightarrow B_2 \Subset B_3$, the following embeddings follow from [10, Theorems 2.1 and 2.2]:

$$L^p(0,T;B_0) \cap W^{\sigma,p}(0,T;B_2) \Subset L^p(0,T;B_1),$$
$$W^{\sigma,p}(0,T;B_2) \Subset \mathcal{C}([0,T];B_3),$$

for $p \in (1,\infty), \sigma \in (0,1)$ and $\sigma p > 1$. We define

$$E_0 := L^\infty(0,T;\mathbb{H}^2) \cap W^{\sigma,p}(0,T;\mathbb{L}^2),$$
$$E := L^2(0,T;\mathbb{H}^1_\rho) \cap \mathcal{C}([0,T];\mathbb{H}^{-1}),$$

for $p \in [2,\infty)$, $\sigma \in (0,\frac{1}{2})$, $\sigma - \frac{1}{p} < \frac{1}{2}$ and $\sigma p > 1$. Since $\mathbb{L}^2 \hookrightarrow \mathbb{L}^2_\rho$, $\mathbb{H}^2 \Subset \mathbb{H}^1_\rho \hookrightarrow \mathbb{L}^2_\rho$ and $\mathbb{H}^1 \Subset \mathbb{L}^2_\rho = (\mathbb{L}^2_\rho)^* \Subset \mathbb{H}^{-1}$, we have

$$E_0 \hookrightarrow L^p(0,T;\mathbb{H}^2) \cap W^{\sigma,p}(0,T;\mathbb{L}^2_\rho) \Subset E.$$

By Lemma 3.2, m_ε takes values in E_0, $\mathbb{P}$-a.s. for any $\varepsilon > 0$. Hence, the set of laws $\{\mathcal{L}(m_\varepsilon)\}$ on E is tight, where for any $\lambda > 0$, $\{|m_\varepsilon|_{E_0} \le \lambda\}$ is compact in E and

$$\mathbb{P}\left(|m_\varepsilon|_{E_0} > \lambda\right) \le \lambda^{-2}\mathbb{E}\left[|m_\varepsilon|^2_{E_0}\right] \le \lambda^{-2}c(R) \to 0, \quad \text{as } \lambda \to \infty.$$

Since E is a separable metric space, by Skorohod theorem there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and a sequence $\{(\tilde{m}_\varepsilon, \tilde{W}_\varepsilon)\}$ of $E \times \mathcal{C}([0,T];H^4(\mathbb{R}^d;\mathbb{R}^d))$-valued random variables defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that

$$\mathcal{L}((m_\varepsilon, W)) = \mathcal{L}((\tilde{m}_\varepsilon, \tilde{W}_\varepsilon)) \quad \text{on } E \times \mathcal{C}([0,T];H^4(\mathbb{R}^d;\mathbb{R}^d)), \quad \forall \varepsilon > 0,$$

and there exists an $E \times \mathcal{C}([0,T];H^4(\mathbb{R}^d;\mathbb{R}^d))$-valued random variable $(\tilde{m}, \tilde{W})$ such that

$$\tilde{m}_\varepsilon \to \tilde{m} \ \text{ in } E, \quad \tilde{W}_\varepsilon \to \tilde{W} \ \text{ in } \mathcal{C}([0,T];H^4(\mathbb{R}^d;\mathbb{R}^d)), \quad \tilde{\mathbb{P}}\text{-a.s.} \tag{4.1}$$

The space $\mathcal{C}([0,T];\mathbb{H}^2)$ is separable and complete. Then by Kuratowski's theorem, $\{\tilde{m}_\varepsilon\}$ have the same laws as $\{m_\varepsilon\}$ on $\mathcal{C}([0,T];\mathbb{H}^2)$, which implies that the estimates in Lemma 3.2 also hold for $\{\tilde{m}_\varepsilon\}$.

We can extend the definition of the $\mathbb{H}^2$-norm to $\mathbb{H}^{-1}$ by setting $|u|_{\mathbb{H}^2} = \infty$ for $u \in \mathbb{H}^{-1} \setminus \mathbb{H}^2$. Then using the pointwise convergence (4.1) in $\mathcal{C}([0,T];\mathbb{H}^{-1})$ and the uniform integrability of $\tilde{m}_\varepsilon$ in $L^{2p}(\tilde{\Omega};\mathcal{C}([0,T];\mathbb{H}^2))$, we deduce that

$$\mathbb{E}\left[|\tilde{m}|^{2p}_{L^\infty(0,T;\mathbb{H}^2)}\right] < \infty, \quad \forall p \in [1,\infty). \tag{4.2}$$

As a result, by (4.1), (A.5) and (A.6), for $p \in [1,\infty)$,

$$\begin{aligned} &\tilde{m}_\varepsilon \to \tilde{m} \quad \text{in } L^{2p}(\tilde{\Omega};L^2(0,T;\mathbb{H}^1_\rho)), \qquad \Delta\tilde{m}_\varepsilon \rightharpoonup \Delta\tilde{m} \quad \text{in } L^{2p}(\tilde{\Omega};L^2(0,T;\mathbb{L}^2_\rho)), \\ &J_\varepsilon\tilde{m}_\varepsilon \to \tilde{m} \quad \text{in } L^{2p}(\tilde{\Omega};L^2(0,T;\mathbb{H}^1_\rho)), \quad \Delta J_\varepsilon\tilde{m}_\varepsilon \rightharpoonup \Delta\tilde{m} \quad \text{in } L^{2p}(\tilde{\Omega};L^2(0,T;\mathbb{L}^2_\rho)). \end{aligned} \tag{4.3}$$

By Gagliardo-Nirenberg's inequality,

$$\left|(\tilde m_\varepsilon-\tilde m)\rho^{\frac12}\right|_{\mathbb{L}^r}\lesssim|\tilde m_\varepsilon-\tilde m|^{1-\theta}_{\mathbb{L}^2_\rho}\left(|\tilde m_\varepsilon|^\theta_{\mathbb{H}^2}+|\tilde m|^\theta_{\mathbb{H}^2}\right),\quad r\in[2,\infty],\tag{4.4}$$

where $\theta=\frac{d}{4}-\frac{d}{2r}\in[0,1)$ for $d=2,3$. The inequality (4.4) still holds with $J_\varepsilon\tilde m_\varepsilon$ in place of $\tilde m_\varepsilon$. Then by the $L^\infty(0,T;\mathbb{H}^2)$ estimates of $\tilde m_\varepsilon$ and $\tilde m$,

$$\begin{aligned}\tilde m_\varepsilon\rho^{\frac12}&\to\tilde m\rho^{\frac12}\quad\text{in }L^{2p}(\tilde\Omega;L^q(0,T;\mathbb{L}^r))\\(J_\varepsilon\tilde m_\varepsilon)\rho^{\frac12}&\to\tilde m\rho^{\frac12}\quad\text{in }L^{2p}(\tilde\Omega;L^q(0,T;\mathbb{L}^r)),\end{aligned}\tag{4.5}$$

for $p\in[1,\infty)$, $q\le\frac{2}{1-\theta}$ and $r\in[2,\infty]$.

4.1. **Vector length of $\tilde m$.** As in Lemma 3.3, let $\varphi(\tilde m):=|1-|\tilde m+e_3|^2|_{\mathbb{L}^2_\rho}\lesssim|\tilde m|^2_{\mathbb{L}^4}+|\tilde m|^2_{\mathbb{L}^2}\in L^{2p}(\tilde\Omega;L^\infty(0,T))$. Then

$$\begin{aligned}&\tilde{\mathbb{E}}\left[\int_0^T|1-|\tilde m(t)+e_3|^2|^2_{\mathbb{L}^2_\rho}\,\mathrm{d}t\right]\\&\lesssim\tilde{\mathbb{E}}\left[\int_0^T\left(\left|1-|\tilde m_\varepsilon(t)+e_3|^2\right|^2_{\mathbb{L}^2_\rho}+\left||\tilde m_\varepsilon(t)+e_3|^2-|\tilde m(t)+e_3|^2\right|^2_{\mathbb{L}^2_\rho}\right)\mathrm{d}t\right]\\&\lesssim\tilde{\mathbb{E}}\left[\int_0^T\left(\left|1-|\tilde m_\varepsilon(t)+e_3|^2\right|^2_{\mathbb{L}^2_\rho}+|\tilde m_\varepsilon(t)-\tilde m(t)|^2_{\mathbb{L}^4_\rho}|\tilde m_\varepsilon(t)+\tilde m(t)|^2_{\mathbb{L}^4_\rho}+|\tilde m_\varepsilon(t)-\tilde m(t)|^2_{\mathbb{L}^2_\rho}\right)\mathrm{d}t\right]\\&\lesssim\tilde{\mathbb{E}}\left[\int_0^T\left|1-|\tilde m_\varepsilon(t)+e_3|^2\right|^2_{\mathbb{L}^2_\rho}\,\mathrm{d}t\right]+\tilde{\mathbb{E}}\left[\sup_{t\in[0,T]}\left(|\tilde m_\varepsilon(t)+\tilde m(t)|^2_{\mathbb{H}^1_\rho}+1\right)\int_0^T|\tilde m_\varepsilon(t)-\tilde m(t)|^2_{\mathbb{H}^1_\rho}\,\mathrm{d}t\right],\end{aligned}\tag{4.6}$$

where the second expectation on the right-hand side converges to 0 by (4.2) and (4.3), thus we only need to focus on the first expectation. Recall the definitions (3.7) and (3.10). For any $q\in[4,\infty)$, the following maps are measurable:

$$\begin{aligned}L^q(0,T;\mathbb{H}^2)\ni u&\mapsto|1-|u+e_3|^2|^2_{\mathbb{L}^2_\rho}\in L^1(0,T),\\L^q(0,T;\mathbb{H}^2)\ni u&\mapsto A_\varepsilon(u)\in L^1(0,T).\end{aligned}$$

Then since $\{\tilde m_\varepsilon\}$ has the same laws as $\{m_\varepsilon\}$ on $L^q(0,T;\mathbb{H}^2)$, it holds by Lemma 3.3 that

$$\begin{aligned}\tilde{\mathbb{E}}\left[\int_0^T|1-|\tilde m_\varepsilon(t)+e_3|^2|^2_{\mathbb{L}^2_\rho}\,\mathrm{d}t\right]&=\mathbb{E}\left[\int_0^T|1-|m_\varepsilon(t)+e_3|^2|^2_{\mathbb{L}^2_\rho}\,\mathrm{d}t\right]\\&\lesssim|1-|J_\varepsilon m_0+e_3|^2|^2_{\mathbb{L}^2_\rho}+\tilde{\mathbb{E}}\left[\int_0^t A_\varepsilon(\tilde m_\varepsilon(s))\,\mathrm{d}s\right]^{\frac12}.\end{aligned}$$

By the assumption that m_0 is bounded in $\mathbb{H}^2$ and the approximation property of J_ε,

$$\begin{aligned}|1-|J_\varepsilon m_0+e_3|^2|^2_{\mathbb{L}^2_\rho}&=\left||m_0+e_3|^2-|J_\varepsilon m_0+e_3|^2\right|^2_{\mathbb{L}^2_\rho}\\&\le|J_\varepsilon m_0-m_0|^2_{\mathbb{L}^2_\rho}|J_\varepsilon m_0+m_0+2e_3|^2_{\mathbb{L}^\infty}\\&\lesssim\left(1+|m_0|^2_{\mathbb{H}^2}\right)|J_\varepsilon m_0-m_0|^2_{\mathbb{L}^2}\ \to0,\quad\text{as }\varepsilon\to0.\end{aligned}$$

For the remainder $A_\varepsilon(\tilde m_\varepsilon)$, we first observe that by (4.3),

$$d_\varepsilon(\tilde m_\varepsilon)=(J_\varepsilon\tilde m_\varepsilon-\tilde m)+(\tilde m-\tilde m_\varepsilon)\to0\quad\text{in }L^{2p}(\tilde\Omega;L^2(0,T;\mathbb{H}^1_\rho)).$$

Similarly,

$$\begin{aligned}\sum_k|d_\varepsilon(\nabla_{g_k}J_\varepsilon\tilde m_\varepsilon)|^2_{\mathbb{L}^2_\rho}&=\sum_k\left|J_\varepsilon\nabla_{g_k}(J_\varepsilon\tilde m_\varepsilon-\tilde m)+\left(J_\varepsilon\nabla_{g_k}\tilde m-\nabla_{g_k}\tilde m\right)+\nabla_{g_k}(\tilde m-J_\varepsilon\tilde m_\varepsilon)\right|^2_{\mathbb{L}^2_\rho}\\&\lesssim\sum_k|g_k|^2_{L^\infty(\mathbb{R}^d;\mathbb{R}^d)}|\nabla(J_\varepsilon\tilde m_\varepsilon-\tilde m)|^2_{\mathbb{L}^2_\rho}+\sum_k\left|J_\varepsilon\nabla_{g_k}\tilde m-\nabla_{g_k}\tilde m\right|^2_{\mathbb{L}^2_\rho},\end{aligned}$$

where the right-hand side converges to 0 in $L^{2p}(\tilde{\Omega};L^1(0,T))$. Also, by the $L^\infty(0,T;\mathbb{H}^2)$-estimate in (4.2), the continuous embedding $\mathbb{H}^2\hookrightarrow\mathbb{L}^\infty$ and Lemma 3.2,

$$\begin{aligned}
&\tilde{\mathbb{E}}\left[\int_0^T \left|(1-|\tilde{m}_\varepsilon+e_3|^2)(\tilde{m}_\varepsilon+e_3)\rho-(1-|\tilde{m}+e_3|^2)(\tilde{m}+e_3)\rho\right|^2_{\mathbb{L}^2}\,\mathrm{d}t\right]\\
&\lesssim \tilde{\mathbb{E}}\left[\int_0^T\left(|\tilde{m}_\varepsilon-\tilde{m}|^2_{\mathbb{L}^2_\rho}+\left|\langle\tilde{m}_\varepsilon-\tilde{m},\tilde{m}_\varepsilon+\tilde{m}+2e_3\rangle(\tilde{m}_\varepsilon+e_3)+|\tilde{m}+e_3|^2(\tilde{m}_\varepsilon-\tilde{m})\right|^2_{\mathbb{L}^2_\rho}\right)\mathrm{d}t\right]\\
&\lesssim \tilde{\mathbb{E}}\left[\int_0^T\left(1+|\tilde{m}_\varepsilon|^4_{\mathbb{L}^\infty}+|\tilde{m}|^4_{\mathbb{L}^\infty}\right)|\tilde{m}_\varepsilon-\tilde{m}|^2_{\mathbb{L}^2_\rho}\,\mathrm{d}t\right]\to 0,\quad \text{as } \varepsilon\to 0,
\end{aligned}$$

which implies $d_\varepsilon((1-|\tilde{m}_\varepsilon+e_3|^2)(\tilde{m}_\varepsilon+e_3)\rho)\to 0$ in $L^{2p}(\tilde{\Omega};L^2(0,T;\mathbb{L}^2))$. Thus for p_ε given in (3.7),

$$\begin{aligned}
\tilde{\mathbb{E}}\left[\int_0^T|p_\varepsilon(\tilde{m}_\varepsilon)|^2_{\mathbb{L}^2}\,\mathrm{d}t\right]&\lesssim\tilde{\mathbb{E}}\left[\int_0^T\left|d_\varepsilon((1-|\tilde{m}_\varepsilon+e_3|^2)(\tilde{m}_\varepsilon+e_3)\rho)\right|^2_{\mathbb{L}^2}\,\mathrm{d}t\right]\\
&\quad+\tilde{\mathbb{E}}\left[\sup_{t\in[0,T]}\left(1+|\tilde{m}_\varepsilon(t)|^4_{\mathbb{L}^\infty}\right)\int_0^T|d_\varepsilon(\tilde{m}_\varepsilon)\rho|^2_{\mathbb{L}^2}\,\mathrm{d}t\right],
\end{aligned}$$

where the right-hand side converges to 0 as $\varepsilon\to 0$. Then

$$\lim_{\varepsilon\to0}\tilde{\mathbb{E}}\left[\int_0^T A_\varepsilon(\tilde{m}_\varepsilon)\,\mathrm{d}t\right]=\lim_{\varepsilon\to0}\tilde{\mathbb{E}}\left[\int_0^T\left(|d_\varepsilon(\tilde{m}_\varepsilon)|^2_{\mathbb{H}^1_\rho}+\sum_k|d_\varepsilon(\nabla_{g_k}J_\varepsilon\tilde{m}_\varepsilon)|^2_{\mathbb{L}^2_\rho}+|p_\varepsilon(\tilde{m}_\varepsilon)|^2_{\mathbb{L}^2}\right)\mathrm{d}t\right]=0.$$

Therefore, the right-hand side of (4.6) converges to 0 as $\varepsilon\to 0$, yielding

$$\mathbb{E}\left[\int_0^T\left|1-|\tilde{m}(t)+e_3|^2\right|^2_{\mathbb{L}^2_\rho}\,\mathrm{d}t\right]\le 0,$$

where $\rho>0$. In other words,

$$|\tilde{m}(t,x)+e_3|=1,\quad \text{a.e.-}(t,x)\in[0,T]\times\mathbb{R}^d,\ \tilde{\mathbb{P}}\text{-a.s.} \tag{4.7}$$

4.2. **Convergence of cut-off function.** Recall (3.2). We define $\tilde{\psi}^{R,(i)}_\varepsilon:=\psi^{(i)}_R(|J_\varepsilon\tilde{m}_\varepsilon+e_3|^2)$ and $\tilde{\psi}^{R,(i)}:=\psi^{(i)}_R(|\tilde{m}+e_3|^2)$, for any non-negative integer i. We simply write $\tilde{\psi}^R_\varepsilon$ and $\tilde{\psi}^R$ when $i=0$. In particular, with $R>1$, by (4.7),

$$\tilde{\psi}^{R,(i)}(t,x)=\psi^{(i)}_R(|\tilde{m}(t,x)+e_3|)=\psi^{(i)}_R(1)=0,\quad\forall i\ge1, \tag{4.8}$$

and thus $\nabla\tilde{\psi}^R(t,x)=\Delta\tilde{\psi}^R(t,x)=0$ for a.e.-$(t,x)\in[0,T]\times\mathbb{R}^d$, $\tilde{\mathbb{P}}$-a.s.

Now we derive some convergence results for $\tilde{\psi}^R_\varepsilon$. By the mean value theorem, we have

$$\begin{aligned}
\left|(\tilde{\psi}^{R,(i)}_\varepsilon-\tilde{\psi}^{R,(i)})\rho^{\frac12}\right|_{\mathbb{L}^r}&\lesssim\left||J_\varepsilon\tilde{m}_\varepsilon+e_3|^2\rho^{\frac12}-|\tilde{m}+e_3|^2\rho^{\frac12}\right|_{\mathbb{L}^r}\\
&\lesssim\left|(J_\varepsilon\tilde{m}_\varepsilon-\tilde{m})\rho^{\frac12}\right|^{1-\theta}_{\mathbb{L}^2}|J_\varepsilon\tilde{m}_\varepsilon-m|^\theta_{\mathbb{H}^2}\,|J_\varepsilon\tilde{m}_\varepsilon+\tilde{m}+2e_3|_{\mathbb{L}^\infty}\\
&\lesssim|J_\varepsilon\tilde{m}_\varepsilon-\tilde{m}|^{1-\theta}_{\mathbb{L}^2_\rho}\left(|\tilde{m}_\varepsilon|^{1+\theta}_{\mathbb{H}^2}+|\tilde{m}|^{1+\theta}_{\mathbb{H}^2}+1\right),
\end{aligned}$$

for any non-negative integer i, where $\theta=\frac{d}{4}-\frac{d}{2r}$ for $r\in[2,\infty]$. Thus, by (4.3),

$$\tilde{\psi}^{R,(i)}_\varepsilon\rho^{\frac12}\to\tilde{\psi}^{R,(i)}\rho^{\frac12}\ \text{ in } L^{2p}(\tilde{\Omega};L^q(0,T;\mathbb{L}^r)), \tag{4.9}$$

for $p\in[1,\infty)$, $q\le\frac{2}{1-\theta}$ and $r\in[2,\infty]$ as in (4.5). In particular, $\tilde{\psi}^R_\varepsilon\to\tilde{\psi}^R$ in $L^{2p}(\tilde{\Omega};L^2(0,T;\mathbb{L}^2_\rho))$. For the gradient of $\tilde{\psi}^R_\varepsilon$, by (3.2) and (4.8),

$$\begin{aligned}
\left|\nabla\left(\tilde{\psi}^R_\varepsilon-\tilde{\psi}^R\right)\right|_{\mathbb{L}^2_\rho}&\lesssim\left|(\tilde{\psi}^{R,(1)}_\varepsilon-\tilde{\psi}^{R,(1)})\rho^{\frac12}\right|_{\mathbb{L}^\infty}|\langle J_\varepsilon\tilde{m}_\varepsilon+e_3,\nabla J_\varepsilon\tilde{m}_\varepsilon\rangle|_{\mathbb{L}^2}\\
&\quad+|\tilde{\psi}^{R,(1)}|_{\mathbb{L}^\infty}|\langle J_\varepsilon\tilde{m}_\varepsilon+e_3,\nabla J_\varepsilon\tilde{m}_\varepsilon\rangle-\langle\tilde{m}+e_3,\nabla\tilde{m}\rangle|_{\mathbb{L}^2_\rho}\\
&\le\left|(\tilde{\psi}^{R,(1)}_\varepsilon-\tilde{\psi}^{R,(1)})\rho^{\frac12}\right|_{\mathbb{L}^\infty}(|\tilde{m}_\varepsilon|_{\mathbb{L}^\infty}+1)\,|\nabla\tilde{m}_\varepsilon|_{\mathbb{L}^2}\\
&\lesssim\left|(\tilde{\psi}^{R,(1)}_\varepsilon-\tilde{\psi}^{R,(1)})\rho^{\frac12}\right|_{\mathbb{L}^\infty}\left(|\nabla\tilde{m}_\varepsilon|^2_{\mathbb{H}^1}+|\nabla\tilde{m}_\varepsilon|_{\mathbb{L}^2}\right),
\end{aligned}$$

where the right-hand side converges to 0 in $L^{2p}(\tilde{\Omega};L^2(0,T))$ since $\tilde{m}_\varepsilon \in L^p(\tilde{\Omega};L^\infty(0,T;\mathbb{H}^2))$ for all $\varepsilon > 0$ and (4.9) holds for $i=1$. Similarly, for the Laplacian of $\tilde{\psi}^R_\varepsilon$,

$$\begin{aligned}
|\Delta(\tilde{\psi}^R_\varepsilon - \tilde{\psi}^R)|_{\mathbb{L}^2_\rho} \lesssim & \left|(\tilde{\psi}^{R,(1)}_\varepsilon - \tilde{\psi}^{R,(1)})\rho^{\frac{1}{2}}\right|_{\mathbb{L}^\infty} \left||\nabla J_\varepsilon \tilde{m}_\varepsilon|^2 + \langle J_\varepsilon \tilde{m}_\varepsilon + e_3, \Delta J_\varepsilon \tilde{m}_\varepsilon\rangle\right|_{\mathbb{L}^2} \\
& + |\tilde{\psi}^{R,(1)}|_{\mathbb{L}^\infty} \left||\nabla J_\varepsilon \tilde{m}_\varepsilon|^2 - |\nabla \tilde{m}|^2 + \langle J_\varepsilon \tilde{m}_\varepsilon + e_3, \Delta J_\varepsilon \tilde{m}_\varepsilon\rangle - \langle \tilde{m} + e_3, \Delta \tilde{m}\rangle\right|_{\mathbb{L}^2_\rho} \\
& + \left|(\tilde{\psi}^{R,(2)}_\varepsilon - \tilde{\psi}^{R,(2)})\rho^{\frac{1}{2}}\right|_{\mathbb{L}^\infty} \left|\langle J_\varepsilon \tilde{m}_\varepsilon + e_3, \nabla J_\varepsilon \tilde{m}_\varepsilon\rangle^2\right|_{\mathbb{L}^2} \\
& + |\tilde{\psi}^{R,(2)}|_{\mathbb{L}^\infty} \left|\langle J_\varepsilon \tilde{m}_\varepsilon + e_3, \nabla J_\varepsilon \tilde{m}_\varepsilon\rangle^2 - \langle \tilde{m} + e_3, \nabla \tilde{m}\rangle^2\right|_{\mathbb{L}^2_\rho} \\
\lesssim & \left|(\tilde{\psi}^{R,(1)}_\varepsilon - \tilde{\psi}^{R,(1)})\rho^{\frac{1}{2}}\right|_{\mathbb{L}^\infty} \left(|\nabla \tilde{m}_\varepsilon|^2_{\mathbb{H}^1} + (|\tilde{m}_\varepsilon|_{\mathbb{L}^\infty} + 1)|\Delta \tilde{m}_\varepsilon|_{\mathbb{L}^2}\right) \\
& + \left|(\tilde{\psi}^{R,(2)}_\varepsilon - \tilde{\psi}^{R,(2)})\rho^{\frac{1}{2}}\right|_{\mathbb{L}^\infty} (|\tilde{m}_\varepsilon|_{\mathbb{L}^\infty} + 1)^2 |\nabla \tilde{m}_\varepsilon|^2_{\mathbb{H}^1},
\end{aligned}$$

where the right-hand side converges to 0 in $L^{2p}(\tilde{\Omega};L^2(0,T))$. Hence, despite the fact that we only have weak convergence for $\Delta\tilde{m}_\varepsilon$ in $\mathbb{L}^2_\rho$, we obtain strong convergence for the cut-off function in $\mathbb{H}^2$:

$$\tilde{\psi}^R_\varepsilon \rho^{\frac{1}{2}} \to \tilde{\psi}^R \rho^{\frac{1}{2}} \text{ in } L^{2p}(\tilde{\Omega};L^2(0,T;\mathbb{H}^2)), \tag{4.10}$$

4.3. **Convergence of drift and diffusion coefficients.** Using results in previous sections, we prove in Lemma 4.1 weak convergences for the drift coefficients F^R_ε and $S_{k,\varepsilon}$, and strong convergence for the diffusion coefficient $G_{k,\varepsilon}$.

Lemma 4.1. *For $p \in [1,\infty)$ and $\varphi \in L^{2p}(\tilde{\Omega};L^4(0,T;C^\infty_c(\mathbb{R}^d)))$,*

$$\lim_{\varepsilon\to 0} \tilde{\mathbb{E}}\left[\left(\int_0^T \left\langle F^R_\varepsilon(\tilde{m}_\varepsilon) - F(\tilde{m}+e_3), \varphi\right\rangle_{\mathbb{L}^2_\rho}(t)\,\mathrm{d}t\right)^p\right] = 0,$$
$$\lim_{\varepsilon\to 0} \tilde{\mathbb{E}}\left[\left(\int_0^T \sum_k \left\langle S_{k,\varepsilon}(\tilde{m}_\varepsilon) - S_k(\tilde{m}), \varphi\right\rangle_{\mathbb{L}^2_\rho}(t)\,\mathrm{d}t\right)^p\right] = 0,$$

and

$$\lim_{\varepsilon\to 0} \tilde{\mathbb{E}}\left[\left(\int_0^T \sum_k |G_{k,\varepsilon}(\tilde{m}_\varepsilon) - G_k(\tilde{m})|^2_{\mathbb{L}^2_\rho}(t)\,\mathrm{d}t\right)^p\right] = 0.$$

Proof. Let $\tilde{u}_\varepsilon := J_\varepsilon \tilde{m}_\varepsilon$. We have

$$\left\langle F^R_\varepsilon(\tilde{m}_\varepsilon) - F(\tilde{m}+e_3), \varphi\right\rangle_{\mathbb{L}^2_\rho} = \left\langle J_\varepsilon\left(\tilde{\psi}^R_\varepsilon \bar{F}(\tilde{u}_\varepsilon + e_3)\right) - \bar{F}(\tilde{m}+e_3) - (J_\varepsilon \nabla_v \tilde{u}_\varepsilon - \nabla_v \tilde{m}), \varphi\rho\right\rangle_{\mathbb{L}^2}$$

where $\tilde{\psi}^R = 1$, a.e.-$(t,x) \in [0,T]\times\mathbb{R}^d$, $\tilde{\mathbb{P}}$-a.s. and

$$\begin{aligned}
\left\langle J_\varepsilon\left(\tilde{\psi}^R_\varepsilon \bar{F}(\tilde{u}_\varepsilon+e_3)\right) - \bar{F}(\tilde{m}+e_3), \varphi\rho\right\rangle_{\mathbb{L}^2} &= \left\langle \bar{F}(\tilde{u}_\varepsilon+e_3), \left(\tilde{\psi}^R_\varepsilon - \tilde{\psi}^R\right)\varphi\rho\right\rangle_{\mathbb{L}^2} \\
&\quad + \left\langle \bar{F}(\tilde{u}_\varepsilon+e_3) - \bar{F}(\tilde{m}+e_3), \varphi\rho\right\rangle_{\mathbb{L}^2} \\
&\quad + \left\langle \tilde{\psi}^R_\varepsilon \bar{F}(\tilde{u}_\varepsilon+e_3), d_\varepsilon(\varphi\rho)\right\rangle_{\mathbb{L}^2} \\
&=: \mathrm{I}_{0,\bar{F}} + \mathrm{I}_{1,\bar{F}} + \mathrm{I}_{2,\bar{F}}.
\end{aligned}$$

Similarly, we define $\{\mathrm{I}_{1,\nabla_v}, \mathrm{I}_{2,\nabla_v}\}, \{\mathrm{I}_{1,S_k}, \mathrm{I}_{2,S_k}\}$ with ∇_v, S_k in place of $\bar{F}$ and 1 in place of $\tilde{\psi}^R_\varepsilon$, for $k \geq 1$.

Convergence of F^R_ε.

(i) For $\mathrm{I}_{0,\bar{F}}$, let $f_\varepsilon = \left(\tilde{\psi}^R_\varepsilon - \tilde{\psi}^R\right)\varphi\rho$. Thus,

$$\begin{aligned}
\langle \bar{F}(\tilde{u}_\varepsilon+e_3), f_\varepsilon\rangle_{\mathbb{L}^2} = & \langle \Delta\tilde{u}_\varepsilon + \alpha(\tilde{u}_\varepsilon+e_3)\times\Delta\tilde{u}_\varepsilon, f_\varepsilon \times (\tilde{u}_\varepsilon+e_3)\rangle_{\mathbb{L}^2} \\
& - h\langle e_3 + \alpha\tilde{u}_\varepsilon \times e_3, f_\varepsilon \times (\tilde{u}_\varepsilon+e_3)\rangle_{\mathbb{L}^2} \\
& + \langle (\tilde{u}_\varepsilon+e_3)\times\Delta\tilde{u}_\varepsilon, \Delta f_\varepsilon\rangle_{\mathbb{L}^2} + 2\langle \Delta\tilde{u}_\varepsilon, \nabla f_\varepsilon \times \nabla\tilde{u}_\varepsilon\rangle_{\mathbb{L}^2} \\
& + \alpha\langle (\tilde{u}_\varepsilon+e_3)\times\Delta\tilde{u}_\varepsilon, f_\varepsilon\times\Delta\tilde{u}_\varepsilon + \Delta f_\varepsilon\times(\tilde{u}_\varepsilon+e_3) + 2\nabla f_\varepsilon\times\nabla\tilde{u}_\varepsilon\rangle_{\mathbb{L}^2}
\end{aligned}$$

$$
\begin{aligned}
&+2\alpha\,\langle \nabla\tilde u_\varepsilon \times \Delta\tilde u_\varepsilon, \nabla f_\varepsilon \times (\tilde u_\varepsilon + e_3) + f_\varepsilon \times \nabla \tilde u_\varepsilon\rangle_{\mathbb{L}^2}\\
&-\gamma\langle (\tilde u_\varepsilon + e_3)\times \nabla_v u, f_\varepsilon\rangle_{\mathbb{L}^2}\\
\le\;& \left(|\tilde u_\varepsilon + e_3|_{\mathbb{L}^\infty} + \alpha|\tilde u_\varepsilon + e_3|^2_{\mathbb{L}^\infty}\right)|\Delta\tilde u_\varepsilon|_{\mathbb{L}^2}|f_\varepsilon|_{\mathbb{L}^2}\\
&+h\left(|\tilde u_\varepsilon|_{\mathbb{L}^2} + \alpha|\tilde u_\varepsilon|^2_{\mathbb{L}^4}\right)|f_\varepsilon|_{\mathbb{L}^2}\\
&+|\tilde u_\varepsilon + e_3|_{\mathbb{L}^\infty}|\Delta m_\varepsilon|_{\mathbb{L}^2}|\Delta f_\varepsilon|_{\mathbb{L}^2} + 2|\Delta\tilde u_\varepsilon|_{\mathbb{L}^2}|\nabla\tilde u_\varepsilon|_{\mathbb{L}^4}|\nabla f_\varepsilon|_{\mathbb{L}^4}\\
&+\alpha|\tilde u_\varepsilon + e_3|_{\mathbb{L}^\infty}|\Delta\tilde u_\varepsilon|^2_{\mathbb{L}^2}|f_\varepsilon|_{\mathbb{L}^\infty} + \alpha|\tilde u_\varepsilon + e_3|^2_{\mathbb{L}^\infty}|\Delta\tilde u_\varepsilon|_{\mathbb{L}^2}|\Delta f_\varepsilon|_{\mathbb{L}^2}\\
&+4\alpha|\tilde u_\varepsilon + e_3|_{\mathbb{L}^\infty}|\Delta\tilde u_\varepsilon|_{\mathbb{L}^2}|\nabla\tilde u_\varepsilon|_{\mathbb{L}^4}|\nabla f_\varepsilon|_{\mathbb{L}^4} + 2\alpha|\Delta\tilde u_\varepsilon|_{\mathbb{L}^2}|\nabla\tilde u_\varepsilon|^2_{\mathbb{L}^4}|f_\varepsilon|_{\mathbb{L}^\infty}\\
&+|\gamma v|_{L^\infty(\mathbb{R}^d;\mathbb{R}^d)}|\tilde u_\varepsilon + e_3|_{\mathbb{L}^\infty}|\nabla\tilde u_\varepsilon|_{\mathbb{L}^2}|f_\varepsilon|_{\mathbb{L}^2}\\
\lesssim\;& \left(1 + |\tilde m_\varepsilon|_{\mathbb{H}^2} + |\tilde m_\varepsilon|^2_{\mathbb{H}^2} + |\tilde m_\varepsilon|^3_{\mathbb{H}^2}\right)|f_\varepsilon|_{\mathbb{H}^2}\\
\lesssim\;& (1 + |\tilde m_\varepsilon|^3_{\mathbb{H}^2})|f_\varepsilon|_{\mathbb{H}^2}.
\end{aligned}
$$

Then by (4.9) with $i=0$, (4.10), and that $\varphi \in L^{2p}(\tilde\Omega; L^2(0,T;\mathbb{H}^2))$,

$$
\begin{aligned}
|f_\varepsilon|_{\mathbb{H}^2} &= \left|\left(\tilde\psi^R_\varepsilon - \tilde\psi^R\right)\varphi\rho\right|_{\mathbb{H}^2}\\
&\lesssim \left|(\tilde\psi^R_\varepsilon - \tilde\psi^R)\rho\right|_{\mathbb{H}^2}|\varphi|_{\mathbb{L}^\infty} + \left|(\tilde\psi^R_\varepsilon - \tilde\psi^R)\rho\right|_{\mathbb{L}^\infty}|\nabla\varphi|_{\mathbb{H}^1} + \left|\nabla\left((\tilde\psi^R_\varepsilon - \tilde\psi^R)\rho\right)\right|_{\mathbb{L}^4}|\nabla\varphi|_{\mathbb{L}^4}\\
&\to 0 \quad \text{in } L^{2p}(\tilde\Omega; L^1(0,T)).
\end{aligned}
$$

This implies

$$
\begin{aligned}
\tilde{\mathbb{E}}\left[\left|\int_0^T \mathrm{I}_{0,\bar F}\,\mathrm{d}t\right|^p\right] &\lesssim \tilde{\mathbb{E}}\left[\left(\int_0^T (1 + |\tilde m_\varepsilon|^3_{\mathbb{H}^2})|f_\varepsilon|_{\mathbb{H}^2}\,\mathrm{d}t\right)^p\right]\\
&\le \tilde{\mathbb{E}}\left[\sup_{t\in[0,T]}\left(1 + |\tilde m_\varepsilon|^{6p}_{\mathbb{H}^2}\right)\right]^{\frac12}\tilde{\mathbb{E}}\left[\left(\int_0^T |f_\varepsilon|_{\mathbb{H}^2}\,\mathrm{d}t\right)^{2p}\right]^{\frac12} \to 0,
\end{aligned}
$$

where the first expectation in the last inequality is finite for all $\varepsilon > 0$ since $\tilde m_\varepsilon$ has the same $L^\infty(0,T;\mathbb{H}^2)$-estimate as m_ε.

(ii) For $\mathrm{I}_{1,\bar F}$, we focus on the difference $\bar F(\tilde u_\varepsilon + e_3) - \bar F(\tilde m + e_3)$:

$$
\begin{aligned}
&\bar F(\tilde u_\varepsilon + e_3) - \bar F(\tilde m + e_3)\\
&= -\left[(\tilde u_\varepsilon - \tilde m)\times(he_3 + \alpha\tilde u_\varepsilon \times he_3) + \alpha(\tilde m + e_3)\times((\tilde u_\varepsilon - \tilde m)\times he_3)\right]\\
&\quad -\gamma\left[(\tilde u_\varepsilon - \tilde m)\times\nabla_v u_\varepsilon + (\tilde m + e_3)\times\nabla_v(\tilde u_\varepsilon - m)\right]\\
&\quad +\left[(\tilde u_\varepsilon - \tilde m)\times(\Delta\tilde u_\varepsilon + \alpha(\tilde u_\varepsilon + e_3)\times\Delta\tilde u_\varepsilon) + \alpha(\tilde m + e_3)\times((\tilde u_\varepsilon - \tilde m)\times\Delta\tilde u_\varepsilon)\right]\\
&\quad +(\tilde m + e_3)\times(\Delta(\tilde u_\varepsilon - \tilde m) + \alpha(\tilde m + e_3)\times\Delta(\tilde u_\varepsilon - \tilde m))\\
&\quad +(\tilde u_\varepsilon - \tilde m)\times\left(\Delta^2\tilde u_\varepsilon + \alpha(\tilde u_\varepsilon + e_3)\times\Delta^2\tilde u_\varepsilon\right)\\
&\quad +\alpha(\tilde m + e_3)\times\left((\tilde u_\varepsilon - \tilde m)\times\Delta^2\tilde u_\varepsilon\right)\\
&\quad +(\tilde m + e_3)\times\left(\Delta^2(\tilde u_\varepsilon - \tilde m) + \alpha(\tilde m + e_3)\times\Delta^2(\tilde u_\varepsilon - \tilde m)\right)\\
&=: \sum_{i=1}^7 \mathrm{H}_i.
\end{aligned}
$$

For $i = 1,2,3$, we deduce $\int_0^T\langle \mathrm{H}_i, \varphi\rho\rangle_{\mathbb{L}^2}\,\mathrm{d}t \to 0$ in $L^p(\tilde\Omega)$ from (4.3), (4.7) and the moment estimates of $\tilde m_\varepsilon, \tilde m$ in $L^{2p}(\tilde\Omega; L^\infty(0,T;\mathbb{H}^2))$.

For $i = 4$, similarly $\int_0^T\langle \mathrm{H}_4, \varphi\rho\rangle_{\mathbb{L}^2}\,\mathrm{d}t \to 0$ in $L^p(\tilde\Omega)$ since $\Delta\tilde m_\varepsilon \rightharpoonup \Delta\tilde m$ in $L^{2p}(\tilde\Omega; L^2(0,T;\mathbb{L}^2_\rho))$ and thanks to (4.7),

$$
\varphi \times (\tilde m + e_3) + \alpha\left(\varphi \times (\tilde m + e_3)\right)\times(\tilde m + e_3) \in L^2(\tilde\Omega; L^2(0,T;\mathbb{L}^2)),
$$

Now we estimate H_5, H_6 and H_7.

$$
\langle \mathrm{H}_5, \varphi\rangle_{\mathbb{L}^2_\rho} = \left\langle (\tilde u_\varepsilon - \tilde m)\times\Delta^2\tilde u_\varepsilon, \varphi\right\rangle_{\mathbb{L}^2_\rho}
$$

$$
\begin{aligned}
&+\alpha\left\langle \tilde{u}_\varepsilon-\tilde{m},\tilde{\psi}^R_\varepsilon\left((\tilde{u}_\varepsilon+e_3)\times\Delta^2\tilde{u}_\varepsilon\right)\times\varphi\right\rangle_{\mathbb{L}^2_\rho}\\
&+\alpha\left\langle \tilde{u}_\varepsilon-\tilde{m},(\tilde{\psi}^R-\tilde{\psi}^R_\varepsilon)\left((\tilde{u}_\varepsilon+e_3)\times\Delta^2\tilde{u}_\varepsilon\right)\times\varphi\right\rangle_{\mathbb{L}^2_\rho}\\
=:&\ \mathrm{H}_{5a}+\alpha\mathrm{H}_{5b}-\alpha\mathrm{H}_{5c}.
\end{aligned}
$$

For H_{5a}, integrating-by-parts,

$$
\begin{aligned}
\mathrm{H}_{5a}&=\langle\Delta(\tilde{u}_\varepsilon-\tilde{m}),\Delta\tilde{m}\times\varphi\rho\rangle_{\mathbb{L}^2}+\langle\tilde{u}_\varepsilon-\tilde{m},\Delta\tilde{u}_\varepsilon\times\Delta(\varphi\rho)\rangle_{\mathbb{L}^2}+2\langle\nabla\tilde{u}_\varepsilon-\tilde{m},\Delta\tilde{u}_\varepsilon\times\nabla(\varphi\rho)\rangle_{\mathbb{L}^2}\\
&\le\langle\Delta(\tilde{u}_\varepsilon-\tilde{m}),\Delta\tilde{m}\times\varphi\rangle_{\mathbb{L}^2_\rho}+c|(\tilde{u}_\varepsilon-\tilde{m})\rho|_{\mathbb{L}^\infty}|\Delta\tilde{u}_\varepsilon|_{\mathbb{L}^2}|\Delta(\varphi\rho)\rho^{-1}|_{\mathbb{L}^2}\\
&\quad+c|\nabla(\tilde{u}_\varepsilon-\tilde{m})\rho|_{\mathbb{L}^4}|\Delta\tilde{u}_\varepsilon|_{\mathbb{L}^2}|\nabla(\varphi\rho)\rho^{-1}|_{\mathbb{L}^4}.
\end{aligned}
$$

where $\Delta\tilde{m}\times\varphi\in L^{2p}(\tilde{\Omega};L^2(0,T;\mathbb{L}^2))$ and

$$
|\nabla(\tilde{u}_\varepsilon-\tilde{m})\rho|_{\mathbb{L}^4}\lesssim|\nabla(\tilde{u}_\varepsilon-\tilde{m})|^{1-\frac{d}{4}}_{\mathbb{L}^2_\rho}\left(|\Delta\tilde{m}_\varepsilon|^{\frac{d}{4}}_{\mathbb{L}^2}+|\Delta\tilde{m}|^{\frac{d}{4}}_{\mathbb{L}^2}\right)\to0\quad\text{in } L^{2p}(\tilde{\Omega};L^4(0,T)).\tag{4.11}
$$

Then, the convergence $\int_0^T\mathrm{H}_{5a}\,\mathrm{d}t\to0$ in $L^p(\tilde{\Omega})$ follows from (2.1), (4.3) and (4.5). Moreover,

$$
\mathrm{H}_{5b}\le|(\tilde{\psi}^R_\varepsilon)^{\frac12}(\tilde{u}_\varepsilon+e_3)\times\Delta^2\tilde{u}_\varepsilon|_{\mathbb{L}^2}|\tilde{u}_\varepsilon-\tilde{m}|_{\mathbb{L}^2_\rho}|(\tilde{\psi}^R_\varepsilon)^{\frac12}\varphi|_{\mathbb{L}^\infty},
$$

where by Lemma 3.2 and (4.3), $\int_0^T\mathrm{H}_{5b}\,\mathrm{d}t\to0$ in $L^p(\tilde{\Omega})$. For H_{5c}, recall $f_\varepsilon=(\psi^R_\varepsilon-\psi^R)\varphi\rho$ from part (i), where $f_\varepsilon\to0\in L^{2p}(\tilde{\Omega};L^1(0,T;\mathbb{H}^2))$. Then,

$$
\begin{aligned}
\mathrm{H}_{5c}&=\left\langle\tilde{u}_\varepsilon-\tilde{m},\left((\tilde{u}_\varepsilon+e_3)\times\Delta^2\tilde{u}_\varepsilon\right)\times f_\varepsilon\right\rangle_{\mathbb{L}^2}\\
&=\langle(\tilde{u}_\varepsilon+e_3)\times\Delta\tilde{u}_\varepsilon,\Delta f_\varepsilon\times(\tilde{u}_\varepsilon-\tilde{m})+f_\varepsilon\times\Delta(\tilde{u}_\varepsilon-\tilde{m})+2\nabla f_\varepsilon\times\nabla(\tilde{u}_\varepsilon-\tilde{m})\rangle_{\mathbb{L}^2}\\
&\quad+2\langle\nabla\tilde{u}_\varepsilon\times\Delta\tilde{u}_\varepsilon,\nabla f_\varepsilon\times(\tilde{u}_\varepsilon-\tilde{m})+f_\varepsilon\times\nabla(\tilde{u}_\varepsilon-\tilde{m})\rangle_{\mathbb{L}^2}\\
&\le(1+|\tilde{u}_\varepsilon|_{\mathbb{L}^\infty})|\Delta\tilde{u}_\varepsilon|_{\mathbb{L}^2}\left(|\tilde{u}_\varepsilon-\tilde{m}|_{\mathbb{L}^\infty}|\Delta f_\varepsilon|_{\mathbb{L}^2}+|\Delta(\tilde{u}_\varepsilon-\tilde{m})|_{\mathbb{L}^2}|f_\varepsilon|_{\mathbb{L}^\infty}+2|\nabla(\tilde{u}_\varepsilon-\tilde{m})|_{\mathbb{L}^4}|\nabla f_\varepsilon|_{\mathbb{L}^4}\right)\\
&\quad+2|\nabla\tilde{u}_\varepsilon|_{\mathbb{L}^4}|\Delta\tilde{u}_\varepsilon|_{\mathbb{L}^2}\left(|\nabla f_\varepsilon|_{\mathbb{L}^4}|\tilde{u}_\varepsilon-\tilde{m}|_{\mathbb{L}^\infty}+|f_\varepsilon|_{\mathbb{L}^\infty}|\nabla(\tilde{u}_\varepsilon-\tilde{m})|_{\mathbb{L}^4}\right).
\end{aligned}
$$

Since $\tilde{u}_\varepsilon,\tilde{m}\in L^{2p}(\tilde{\Omega};L^\infty(0,T;\mathbb{H}^2))$, $\int_0^T\mathrm{H}_{5c}\,\mathrm{d}t\to0$ in $L^p(\tilde{\Omega})$ by the convergence of f_ε as in part (i).

For H_6, recall (4.7). Then,

$$
\begin{aligned}
\alpha^{-1}\langle\mathrm{H}_6,\varphi\rho\rangle_{\mathbb{L}^2}&=\left\langle\Delta^2\tilde{u}_\varepsilon,(\varphi\rho\times(\tilde{m}+e_3))\times(\tilde{u}_\varepsilon-\tilde{m})\right\rangle_{\mathbb{L}^2}\\
&=\langle(\tilde{u}_\varepsilon-\tilde{m}),\Delta\tilde{u}_\varepsilon\times(\Delta(\varphi\rho)\times(\tilde{m}+e_3)+2\nabla(\varphi\rho)\times\nabla\tilde{m})\rangle_{\mathbb{L}^2}\\
&\quad+\langle(\tilde{u}_\varepsilon-\tilde{m}),\Delta\tilde{u}_\varepsilon\times(\varphi\times\Delta\tilde{m})\rangle_{\mathbb{L}^2_\rho}\\
&\quad+\langle\Delta(\tilde{u}_\varepsilon-\tilde{m}),\Delta\tilde{m}\times(\varphi\times(\tilde{m}+e_3))\rangle_{\mathbb{L}^2_\rho}\\
&\quad+2\langle\nabla(\tilde{u}_\varepsilon-\tilde{m}),\Delta\tilde{u}_\varepsilon\times(\nabla(\varphi\rho)\times(\tilde{m}+e_3)+\varphi\rho\times\nabla\tilde{m})\rangle_{\mathbb{L}^2}\\
&\lesssim|(\tilde{u}_\varepsilon-\tilde{m})\rho|_{\mathbb{L}^\infty}|\Delta\tilde{u}_\varepsilon|_{\mathbb{L}^2}\left(|\Delta(\varphi\rho)\rho^{-1}|_{\mathbb{L}^2}+|\nabla\tilde{m}|_{\mathbb{L}^4}|\nabla(\varphi\rho)\rho^{-1}|_{\mathbb{L}^4}\right)\\
&\quad+|(\tilde{u}_\varepsilon-\tilde{m})\rho|_{\mathbb{L}^\infty}|\Delta\tilde{u}_\varepsilon|_{\mathbb{L}^2}|\Delta\tilde{m}|_{\mathbb{L}^2}|\varphi|_{\mathbb{H}^2}\\
&\quad+\langle\Delta(\tilde{u}_\varepsilon-\tilde{m}),\Delta\tilde{m}\times(\varphi\times(\tilde{m}+e_3))\rangle_{\mathbb{L}^2_\rho}\\
&\quad+|\nabla(\tilde{u}_\varepsilon-\tilde{m})\rho|_{\mathbb{L}^4}|\Delta\tilde{u}_\varepsilon|_{\mathbb{L}^2}\left(|\nabla(\varphi\rho)\rho^{-1}|_{\mathbb{L}^4}+|\nabla\tilde{m}|_{\mathbb{L}^4}|\varphi|_{\mathbb{H}^2}\right),
\end{aligned}
$$

where the third term on the right-hand side converges by the weak convergence in (4.3) and the fact $\Delta\tilde{m}\times(\varphi\times(\tilde{m}+e_3))\in L^{2p}(\tilde{\Omega};L^2(0,T;\mathbb{L}^2))$. Thus, $\int_0^T\langle\mathrm{H}_6,\varphi\rho\rangle_{\mathbb{L}^2}\,\mathrm{d}t\to0$ in $L^p(\tilde{\Omega})$ holds by (2.1), (4.5) and (4.11).

For H_7, we have

$$
\begin{aligned}
\langle\mathrm{H}_7,\varphi\rho\rangle_{\mathbb{L}^2}&=\left\langle\Delta^2(\tilde{u}_\varepsilon-\tilde{m}),\varphi\rho\times(\tilde{m}+e_3)+\alpha(\tilde{m}+e_3)\times((\tilde{m}+e_3)\times\varphi\rho)\right\rangle_{\mathbb{L}^2}\\
&=\left\langle\Delta(\tilde{u}_\varepsilon-\tilde{m}),\Delta\left(\varphi\rho\times(\tilde{m}+e_3)+\alpha(\tilde{m}+e_3)\times((\tilde{m}+e_3)\times\varphi\rho)\right)\rho^{-1}\right\rangle_{\mathbb{L}^2_\rho},
\end{aligned}
$$

where $\Delta(\varphi\rho\times(\tilde{m}+e_3)+\alpha(\tilde{m}+e_3)\times((\tilde{m}+e_3)\times\varphi\rho))\rho^{-1}\in L^2(\tilde{\Omega};L^2(0,T;\mathbb{L}^2))$. Then $\int_0^T\langle\mathrm{H}_7,\varphi\rho\rangle_{\mathbb{L}^2}\,\mathrm{d}t\to0$ in $L^p(\tilde{\Omega})$ follows from the weak convergence in (4.3) .

Therefore, as $\varepsilon \to 0$, for $p \in [1,\infty)$,

$$\tilde{\mathbb{E}}\left[\left|\int_0^T \mathrm{I}_{1,\bar{F}}\,\mathrm{d}t\right|^p\right] \lesssim \sum_{i=1}^{7} \tilde{\mathbb{E}}\left[\left|\int_0^T \langle \mathrm{H}_i, \varphi\rho\rangle_{\mathbb{L}^2}\,\mathrm{d}t\right|^p\right] \to 0.$$

(iii) For $\mathrm{I}_{2,\bar{F}}$, recall that for any $p \in [1,\infty)$,

$$\sup_{\varepsilon>0} \tilde{\mathbb{E}}\left[\left(\int_0^T \left|(\tilde{\psi}_\varepsilon^R)^{\frac{1}{2}}(\tilde{u}_\varepsilon + e_3)\times\Delta^2\tilde{u}_\varepsilon\right|_{\mathbb{L}^2}^2(t)\,\mathrm{d}t\right)^p\right] < \infty.$$

Then it is clear from (4.2) that

$$\sup_{\varepsilon>0} \tilde{\mathbb{E}}\left[\left(\int_0^T \left|\tilde{\psi}_\varepsilon^R \bar{F}(\tilde{u}_\varepsilon + e_3)\right|_{\mathbb{L}^2}^2(t)\,\mathrm{d}t\right)^p\right] < \infty.$$

Since $\mathrm{I}_{2,\bar{F}} \le |\tilde{\psi}_\varepsilon^R \bar{F}(\tilde{u}_\varepsilon+e_3)|_{\mathbb{L}^2} |d_\varepsilon(\varphi\rho)|_{\mathbb{L}^2}$, by the approximation property of J_ε, we have $d_\varepsilon(\varphi\rho) \to 0$ in $L^{2p}(\tilde{\Omega}; L^2(0,T;\mathbb{L}^2))$ and thus $\int_0^T \mathrm{I}_{2,\bar{F}}\,\mathrm{d}t \to 0$ in $L^p(\tilde{\Omega})$.

Moreover, since $\tilde{\psi}_\varepsilon^R(\tilde{u}_\varepsilon+e_3)\times\Delta^2\tilde{u}_\varepsilon$ and $\tilde{\psi}_\varepsilon^R(\tilde{u}_\varepsilon+e_3)\times((\tilde{u}_\varepsilon+e_3)\times\Delta^2\tilde{u}_\varepsilon)$ are in $L^{2p}(\tilde{\Omega}; L^2(0,T;\mathbb{L}^2))$, there exist measurable processes $Y, Z \in L^{2p}(\tilde{\Omega}; L^2(0,T;\mathbb{L}^2))$ such that

$$\begin{aligned}
\tilde{\psi}_\varepsilon^R(\tilde{u}_\varepsilon+e_3)\times\Delta^2\tilde{u}_\varepsilon &\rightharpoonup Y \quad \text{in } L^{2p}(\tilde{\Omega}; L^2(0,T;\mathbb{L}^2)),\\
\tilde{\psi}_\varepsilon^R(\tilde{u}_\varepsilon+e_3)\times\big((\tilde{u}_\varepsilon+e_3)\times\Delta^2\tilde{u}_\varepsilon\big) &\rightharpoonup Z \quad \text{in } L^{2p}(\tilde{\Omega}; L^2(0,T;\mathbb{L}^2)).
\end{aligned}$$

By the arguments in previous steps and the uniqueness of weak limit, we have $Y = (\tilde{m}+e_3)\times\Delta^2\tilde{m}$ and $Z = (\tilde{m}+e_3)\times((\tilde{m}+e_3)\times\Delta^2\tilde{m})$ in $L^{2p}(\tilde{\Omega}; L^2(0,T;\mathbb{L}^2))$ in the weak sense (2.4).

(iv) Similar calculations hold for I_{1,∇_v} and I_{2,∇_v}. By the uniform integrability and strong convergence of $\tilde{m}_\varepsilon$ in $L^{2p}(\tilde{\Omega}; L^2(0,T;\mathbb{H}^1_\rho))$ and properties of J_ε,

$$\begin{aligned}
&\tilde{\mathbb{E}}\left[\int_0^T \langle \nabla_v(\tilde{u}_\varepsilon - \tilde{m}), \varphi\rangle_{\mathbb{L}^2_\rho}\,\mathrm{d}t\right] \le \tilde{\mathbb{E}}\left[\int_0^T |v|_{L^\infty(\mathbb{R}^d;\mathbb{R}^d)} |\nabla(\tilde{u}_\varepsilon-\tilde{m})|_{\mathbb{L}^2_\rho} |\varphi|_{\mathbb{L}^2}\,\mathrm{d}t\right] \to 0,\\
&\tilde{\mathbb{E}}\left[\int_0^T \langle \nabla_v \tilde{u}_\varepsilon, d_\varepsilon(\varphi\rho)\rangle_{\mathbb{L}^2}\,\mathrm{d}t\right] \le |v|_{L^\infty(\mathbb{R}^d;\mathbb{R}^d)} \left(\sup_{\varepsilon>0}\tilde{\mathbb{E}}\left[|\nabla\tilde{u}_\varepsilon|^2_{L^\infty(0,T;\mathbb{L}^2)}\right]\right)^{\frac{1}{2}} \tilde{\mathbb{E}}\left[\int_0^T |d_\varepsilon(\varphi\rho)|^2_{\mathbb{L}^2}\,\mathrm{d}t\right]^{\frac{1}{2}} \to 0,
\end{aligned}$$

as $\varepsilon \to 0$. The same arguments follow for convergence in $L^p(\tilde{\Omega})$, $p \in (1,\infty)$.

Convergence of $S_{k,\varepsilon}$.

For I_{1,S_k},

$$\begin{aligned}
\mathrm{I}_{1,S_k} &= \langle S_k(\tilde{u}_\varepsilon) - S_k(\tilde{m}), \varphi\rangle_{\mathbb{L}^2_\rho}\\
&= \big\langle (\nabla_{g_k})^2(\tilde{u}_\varepsilon - \tilde{m}), \varphi\rho\big\rangle_{\mathbb{L}^2}\\
&= -\big\langle \nabla_{g_k}(\tilde{u}_\varepsilon-\tilde{m}), \nabla_{g_k}(\varphi\rho) + (\mathrm{div} g_k)\varphi\rho\big\rangle_{\mathbb{L}^2}\\
&\lesssim |g_k|_{W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)} |\nabla(\tilde{u}_\varepsilon-\tilde{m})\rho|_{\mathbb{L}^2} \left(|\nabla(\varphi\rho)\rho^{-1}|_{\mathbb{L}^2} + |\varphi|_{\mathbb{L}^2}\right).
\end{aligned}$$

Then $\sum_k \int_0^T \mathrm{I}_{1,S_k}\,\mathrm{d}t \to 0$ in $L^p(\tilde{\Omega})$ by (2.1), (2.2) and (4.3).

As in the case of $\mathrm{I}_{2,\bar{F}}$, the convergence of $\sum_k \mathrm{I}_{2,S_k} = \sum_k \langle S_k(\tilde{u}_\varepsilon), d_\varepsilon(\varphi\rho)\rangle_{\mathbb{L}^2}$ follows from (2.2), the $L^2(0,T;\mathbb{H}^2)$-integrability of $\tilde{u}_\varepsilon$ and properties of J_ε.

Convergence of $G_{k,\varepsilon}$.

By (A.5),

$$\begin{aligned}
|G_{k,\varepsilon}(\tilde{m}_\varepsilon) - G_k(\tilde{m})|_{\mathbb{L}^2_\rho} &\le |J_\varepsilon(G_k(\tilde{u}_\varepsilon) - G_k(\tilde{m}))|_{\mathbb{L}^2_\rho} + |J_\varepsilon G_k(\tilde{m}) - G_k(\tilde{m}))|_{\mathbb{L}^2_\rho}\\
&\le c|G_k(\tilde{u}_\varepsilon) - G_k(\tilde{m})|_{\mathbb{L}^2_\rho} + |d_\varepsilon(G_k(\tilde{m})) - G_k(\tilde{m}))|_{\mathbb{L}^2}.
\end{aligned}$$

Since $G_k(\tilde{m}) = -\nabla_{g_k}\tilde{m} \in L^{2p}(\tilde{\Omega}; L^\infty(0,T;\mathbb{L}^2))$, and

$$|G_k(\tilde{u}_\varepsilon) - G_k(\tilde{m})|_{\mathbb{L}^2_\rho} \le |g_k|_{L^\infty(\mathbb{R}^d;\mathbb{R}^d)} |\nabla(\tilde{u}_\varepsilon - \tilde{m})|_{\mathbb{L}^2_\rho},$$

we obtain convergence for $\sum_k |G_{k,\varepsilon}(\tilde{m}_\varepsilon) - G_k(\tilde{m})|^2_{\mathbb{L}^2_\rho}$ in $L^p(\tilde{\Omega}; L^1(0,T))$ from (2.2), (4.3) and approximation properties of J_ε. □

By the density of $L^4(0,T;\mathcal{C}_c^\infty(\mathbb{R}^d))$ in $L^2(0,T;\mathbb{L}^2_\rho)$, the weak convergences in Lemma 4.1 reduce to

$$F^R_\varepsilon(\tilde{m}_\varepsilon) + \frac{1}{2}\sum_k S_{k,\varepsilon}(\tilde{m}_\varepsilon) \rightharpoonup F(\tilde{m}+e_3) + \frac{1}{2}\sum_k S_k(\tilde{m}) \quad \text{in } L^{2p}(\tilde{\Omega}; L^2(0,T;\mathbb{L}^2_\rho)).$$

5. Proof of Theorem 2.1

5.1. **Existence of martingale solution.** As in [2, Section 5], we can show that the limit process $(\tilde{m}, \tilde{W})$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ from Section 4 is a martingale solution of the equation

$$\tilde{m}(t) = \tilde{m}_0 + \int_0^t \bar{F}(\tilde{m}+e_3)(s)\,\mathrm{d}s + \frac{1}{2}\sum_k \int_0^t S_k(\tilde{m})(s)\,\mathrm{d}s + \sum_k \int_0^t G_k(\tilde{m})(s)\,\mathrm{d}\tilde{W}_k(s). \tag{5.1}$$

We only outline the arguments here. Since $\{\tilde{W}_\varepsilon\}$ have the same laws as W, the processes $\tilde{W}_\varepsilon$ and thus $\tilde{W}$ are Wiener processes with covariance Q on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Using the pointwise convergence $(\tilde{m}_\varepsilon, \tilde{W}_\varepsilon) \to (\tilde{m}, \tilde{W})$ in $E \times \mathcal{C}([0,T]; H^4(\mathbb{R}^d;\mathbb{R}^d))$, Lemma 4.1 and the embedding $\mathbb{L}^2_\rho \Subset \mathbb{H}^{-1}$, we deduce from the uniqueness of weak limit in $L^2(\tilde{\Omega};\mathbb{H}^{-1})$ that $\tilde{m}(t)$ satisfies (5.1) (in the weak sense) for every $t \in [0,T]$ and

$$\mathbb{E}\left[|\tilde{m}(t)|^2_{L^\infty(0,T;\mathbb{H}^2)} + \left|(\tilde{m}+e_3)\times\Delta^2\tilde{m}\right|^2_{L^2(0,T;\mathbb{L}^2)}\right] < \infty,$$

with $|\tilde{m}+e_3| = 1$, a.e. in $[0,T]\times\mathbb{R}^d$, $\tilde{\mathbb{P}}$-a.s. Thus, $\tilde{M} := \tilde{m}+e_3$ is a solution of (2.3) in the sense of Definition 2.1.

Using Kolmogorov's criterion, we deduce that $\tilde{m} = \tilde{M} - e_3$ has paths in $\mathcal{C}^\sigma([0,T];\mathbb{L}^2)$ $\tilde{\mathbb{P}}$-a.s. Let $0 \le s \le t \le T$ and $p \in [1,\infty)$, we have

$$\begin{aligned}
\tilde{\mathbb{E}}\left[\left(\int_s^t |F(\tilde{M}(r))|_{\mathbb{L}^2}\,\mathrm{d}r\right)^{2p}\right] &\le |t-s|^p\tilde{\mathbb{E}}\left[\left(\int_s^t |F(\tilde{M}(r))|^2_{\mathbb{L}^2}\,\mathrm{d}r\right)^{p}\right] \\
&\lesssim |t-s|^p\tilde{\mathbb{E}}\left[\left(\int_0^T \left(|\nabla\tilde{m}|^2_{\mathbb{H}^1} + |(\tilde{m}+e_3)\times\Delta^2\tilde{m}|^2_{\mathbb{L}^2}\right)(r)\,\mathrm{d}r\right)^{p}\right] \\
&\lesssim |t-s|^p, \\
\tilde{\mathbb{E}}\left[\left(\sum_k\int_s^t |S_k(\tilde{m}(r))|_{\mathbb{L}^2}\,\mathrm{d}r\right)^{2p}\right] &\le \tilde{\mathbb{E}}\left[\left(\sum_k |g_k|^2_{W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)}\int_s^t |\nabla\tilde{m}(r)|_{\mathbb{H}^1}\,\mathrm{d}r\right)^{2p}\right] \\
&\lesssim |t-s|^{2p}\tilde{\mathbb{E}}\left[|\nabla\tilde{m}|^{2p}_{L^\infty(0,T;\mathbb{H}^1)}\right] \\
&\lesssim |t-s|^{2p}, \\
\tilde{\mathbb{E}}\left[\left(\sum_k\left|\int_s^t G_k(\tilde{m}(r))\,\mathrm{d}\tilde{W}_k(r)\right|_{\mathbb{L}^2}\right)^{2p}\right] &= \tilde{\mathbb{E}}\left[\left(\sum_k\int_s^t |G_k(\tilde{m}(r))|^2_{\mathbb{L}^2}\,\mathrm{d}r\right)^{p}\right] \\
&\le \tilde{\mathbb{E}}\left[\left(\sum_k |g_k|^2_{L^\infty(\mathbb{R}^d;\mathbb{R}^d)}\int_s^t |\tilde{m}(r)|^2_{\mathbb{H}^1}\,\mathrm{d}r\right)^{p}\right] \\
&\lesssim |t-s|^p.
\end{aligned}$$

Therefore,

$$\tilde{\mathbb{E}}\left[|\tilde{m}(t)-\tilde{m}(s)|^{2p}_{\mathbb{L}^2}\right] \lesssim |t-s|^p,$$

proving the $\mathcal{C}^\sigma([0,T];\mathbb{L}^2)$-regularity for $\sigma \in [0,\frac{1}{2})$.

5.2. **Pathwise uniqueness.** Let M_1 and M_2 be two martingale solutions of the equation (2.3) defined on the same filtered probability space and with the same Wiener process $(\Omega,\mathcal{F},(\mathcal{F}_t),\mathbb{P},W)$. As usual, we set $m_1 := M_1 - e_3$ and $m_2 := M_2 - e_3$. Then, m_1 and m_2 are martingale solutions of (5.1). Note that in distributional sense

$$M_1\times(M_1\times\Delta M_1) = -|\nabla M_1|^2 M_1 - \Delta M_1,$$
$$M_1\times(M_1\times\Delta^2 M_1) = \left\langle M_1,\Delta^2 M_1\right\rangle M_1 - \Delta^2 M_1.$$

Let $y := M_1 - M_2 = m_1 - m_2 \in \mathbb{L}^2$. By the Gagliardo-Nirenberg's inequality,

$$\begin{aligned} |\nabla y|_{\mathbb{L}^2}^2 &= -\langle y,\Delta y\rangle_{\mathbb{L}^2} \le |y|_{\mathbb{L}^2}|\Delta y|_{\mathbb{L}^2}\\ |\nabla y|_{\mathbb{L}^4} &\le c|y|_{\mathbb{L}^2}^{\frac12-\frac d8}|\Delta y|_{\mathbb{L}^2}^{\frac12+\frac d8},\\ |y|_{\mathbb{L}^\infty} &\le c|y|_{\mathbb{L}^2}^{1-\frac d4}|\Delta y|_{\mathbb{L}^2}^{\frac d4}. \end{aligned} \tag{5.2}$$

In addition, $|y|_{\mathbb{L}^\infty}\le 2$, $y(0)=0$ and we have

$$\begin{aligned} \mathrm{d}y &= y\times(\Delta M_1+\Delta^2 M_1)\,\mathrm{d}t + M_2\times(\Delta y+\Delta^2 y)\,\mathrm{d}t - y\times he_3\,\mathrm{d}t\\ &\quad -\alpha\left(\langle\nabla y,\nabla(M_1+M_2)\rangle M_1 + |\nabla M_2|^2 y\right)\,\mathrm{d}t\\ &\quad +\alpha\left(\left\langle y,\Delta^2 M_1\right\rangle M_1 + \left\langle M_2,\Delta^2 y\right\rangle M_1 + \left\langle M_2,\Delta^2 M_2\right\rangle y\right)\,\mathrm{d}t\\ &\quad -\alpha\left(y\times(M_1\times he_3) + M_2\times(y\times he_3)\right)\,\mathrm{d}t\\ &\quad -\gamma(y\times\nabla_v M_1 + M_2\times\nabla_v y)\,\mathrm{d}t\\ &\quad -\alpha\left(\Delta^2 y+\Delta y\right)\,\mathrm{d}t - \nabla_v y\,\mathrm{d}t + \frac12\sum_k(\nabla_{g_k})^2 y\,\mathrm{d}t - \sum_k\nabla_{g_k}y\,\mathrm{d}W_k. \end{aligned}$$

Applying Itô's lemma to $\frac12|y(t)|_{\mathbb{L}^2}^2$,

$$\begin{aligned} \frac12\mathrm{d}|y(t)|_{\mathbb{L}^2}^2 &= \left\langle y, M_2\times(\Delta y+\Delta^2 y) - \alpha M_2\times(y\times he_3)\right\rangle_{\mathbb{L}^2}\,\mathrm{d}t\\ &\quad -\alpha\left\langle y,\langle\nabla y,\nabla(M_1+M_2)\rangle M_1\right\rangle_{\mathbb{L}^2}\,\mathrm{d}t - \alpha\int_{\mathbb{R}^d}|\nabla M_2|^2|y|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\quad +\alpha\left\langle y,\left\langle y,\Delta^2 M_1\right\rangle M_1 + \left\langle M_2,\Delta^2 y\right\rangle M_1 + \left\langle M_2,\Delta^2 M_2\right\rangle y\right\rangle_{\mathbb{L}^2}\,\mathrm{d}t\\ &\quad -\gamma\langle y, M_2\times\nabla_v y\rangle_{\mathbb{L}^2}\,\mathrm{d}t + \alpha\left(|\nabla y|_{\mathbb{L}^2}^2 - |\Delta y|_{\mathbb{L}^2}^2\right)\,\mathrm{d}t\\ &\quad +\frac12\sum_k\left(\left\langle y,(\nabla_{g_k})^2 y\right\rangle_{\mathbb{L}^2} + |\nabla_{g_k}y|_{\mathbb{L}^2}^2\right)\,\mathrm{d}t - \sum_k\left\langle y,\nabla_{g_k}y\right\rangle_{\mathbb{L}^2}\,\mathrm{d}W_k, \end{aligned}$$

where $\langle y,\nabla_v y\rangle_{\mathbb{L}^2}=0$ and by (5.2),

$$|\nabla y|_{\mathbb{L}^2}^2 \le \delta^{-1}|y|_{\mathbb{L}^2}^2 + \delta|\Delta y|_{\mathbb{L}^2}^2,$$

for $\delta\in(0,1)$.

For the Stratonovich and Itô correction terms,

$$\begin{aligned} \sum_k\left(\left\langle y,(\nabla_{g_k})^2 y\right\rangle_{\mathbb{L}^2} + |\nabla_{g_k}y|_{\mathbb{L}^2}^2\right) &= -\sum_k\left\langle y,(\mathrm{div}g_k)\nabla_{g_k}y\right\rangle_{\mathbb{L}^2}\\ &\lesssim \delta^{-1}\sum_k|g_k|_{W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)}^2|y|_{\mathbb{L}^2}^2 + \delta|\nabla y|_{\mathbb{L}^2}^2\\ &\lesssim |y|_{\mathbb{L}^2}^2 + \delta|\Delta y|_{\mathbb{L}^2}^2. \end{aligned}$$

Also, for the diffusion term,

$$\begin{aligned} \sum_k\left\langle y,\nabla_{g_k}y\right\rangle_{\mathbb{L}^2}^2 &= \frac12\sum_k\left\langle y,(\mathrm{div}g_k)y\right\rangle_{\mathbb{L}^2}^2\\ &\le \frac12\sum_k|\mathrm{div}g_k|_{\mathbb{L}^2}^2|y|_{\mathbb{L}^\infty}^2|y|_{\mathbb{L}^2}^2 \end{aligned}$$

$$\leq 2\sum_k |g_k|^2_{H^1(\mathbb{R}^d;\mathbb{R}^d)}|y|^2_{\mathbb{L}^2},$$

where the right-hand side has finite expectation, implying that the Itô integral is a well-defined continuous martingale.

We estimate the main drift part below.

$$\begin{aligned}
&\langle y, M_2\times\Delta y - \alpha M_2\times(y\times he_3) - \gamma M_2\times\nabla_v y\rangle_{\mathbb{L}^2}\\
&= \langle y, M_2\times\Delta y\rangle_{\mathbb{L}^2} - \alpha\langle y, M_2\times(y\times he_3)\rangle_{\mathbb{L}^2} - \gamma\langle y, M_2\times\nabla_v y\rangle_{\mathbb{L}^2}\\
&\lesssim |y|^2_{\mathbb{L}^2} + \delta|\Delta y|^2_{\mathbb{L}^2},
\end{aligned}$$

and

$$\begin{aligned}
\left\langle y, M_2\times\Delta^2 y\right\rangle_{\mathbb{L}^2} &= \langle y\times\Delta M_2, \Delta y\rangle_{\mathbb{L}^2} + 2\langle\nabla y\times\nabla M_2, \Delta y\rangle_{\mathbb{L}^2}\\
&\leq |y|_{\mathbb{L}^\infty}|\Delta M_2|_{\mathbb{L}^2}|\Delta y|_{\mathbb{L}^2} + 2|\nabla y|_{\mathbb{L}^4}|\nabla M_2|_{\mathbb{L}^4}|\Delta y|_{\mathbb{L}^2}\\
&\lesssim |\Delta M_2|_{\mathbb{L}^2}|y|^{1-\frac{d}{4}}_{\mathbb{L}^2}|\Delta y|^{1+\frac{d}{4}}_{\mathbb{L}^2} + |\nabla M_2|_{\mathbb{H}^1}|y|^{\frac{1}{2}-\frac{d}{8}}_{\mathbb{L}^2}|\Delta y|^{\frac{3}{2}+\frac{d}{8}}_{\mathbb{L}^2}\\
&\lesssim \left(|\Delta M_2|^{\frac{8}{4-d}}_{\mathbb{L}^2} + |\Delta M_2|^2_{\mathbb{L}^2} + |\nabla M_2|^{\frac{16}{4-d}}_{\mathbb{H}^2}\right)|y|^2_{\mathbb{L}^2} + \delta|\Delta y|^2_{\mathbb{L}^2}.
\end{aligned}$$

Similarly, for the α terms,

$$\begin{aligned}
\langle y, \langle\nabla y, \nabla(M_1+M_2)\rangle M_1\rangle_{\mathbb{L}^2} &\leq |y|_{\mathbb{L}^\infty}|\nabla y|_{\mathbb{L}^2}|\nabla(M_1+M_2)|_{\mathbb{L}^2}\\
&\lesssim |y|_{\mathbb{L}^\infty}|y|^{\frac{1}{2}}_{\mathbb{L}^2}|\Delta y|^{\frac{1}{2}}_{\mathbb{L}^2}|\nabla(M_1+M_2)|_{\mathbb{L}^2}\\
&\lesssim |y|^{\frac{3}{2}-\frac{d}{4}}_{\mathbb{L}^2}|\Delta y|^{\frac{1}{2}+\frac{d}{4}}_{\mathbb{L}^2}|\nabla(M_1+M_2)|_{\mathbb{L}^2}\\
&\lesssim |\nabla(M_1+M_2)|^{\frac{8}{6-d}}_{\mathbb{L}^2}|y|^2_{\mathbb{L}^2} + \delta|\Delta y|^2_{\mathbb{L}^2},
\end{aligned}$$

and

$$\begin{aligned}
\left\langle y, \left\langle y, \Delta^2 M_1\right\rangle M_1\right\rangle_{\mathbb{L}^2} &= \left\langle (y\cdot M_1)y, \Delta^2 M_1\right\rangle_{\mathbb{L}^2}\\
&= \langle(\Delta y\cdot M_1)y + 2(\nabla y\cdot\nabla M_1)y + (y\cdot M_1)\Delta y, \Delta M_1\rangle_{\mathbb{L}^2}\\
&\quad + 2\langle(\nabla y\cdot M_1 + y\cdot\nabla M_1)\nabla y, \Delta M_1\rangle_{\mathbb{L}^2} + \int_{\mathbb{R}^d}\langle y, \Delta M_1\rangle^2\,\mathrm{d}x\\
&\leq 2\left(|y|_{\mathbb{L}^\infty}|\Delta y|_{\mathbb{L}^2} + |\nabla y|^2_{\mathbb{L}^4}\right)|\Delta M_1|_{\mathbb{L}^2}\\
&\quad + 4|y|_{\mathbb{L}^\infty}|\nabla y|_{\mathbb{L}^4}|\nabla M_1|_{\mathbb{L}^4}|\Delta M_1|_{\mathbb{L}^2} + |y|^2_{\mathbb{L}^\infty}|\Delta M_1|^2_{\mathbb{L}^2}\\
&\lesssim |y|^{1-\frac{d}{4}}_{\mathbb{L}^2}|\Delta y|^{1+\frac{d}{4}}_{\mathbb{L}^2}|\Delta M_1|_{\mathbb{L}^2} + \left(|y|^{\frac{3}{2}-\frac{3d}{8}}_{\mathbb{L}^2}|\Delta y|^{\frac{1}{2}+\frac{3d}{8}}_{\mathbb{L}^2} + |y|^{2-\frac{d}{2}}_{\mathbb{L}^2}|\Delta y|^{\frac{d}{2}}_{\mathbb{L}^2}\right)|\nabla M_1|^2_{\mathbb{H}^1}\\
&\lesssim \left(|\Delta M_1|^{\frac{8}{4-d}}_{\mathbb{L}^2} + |\nabla M_1|^{\frac{32}{12-3d}}_{\mathbb{H}^1}\right)|y|^2_{\mathbb{L}^2} + \delta|\Delta y|^2_{\mathbb{L}^2},\\
\left\langle y, \left\langle M_2, \Delta^2 M_2\right\rangle y\right\rangle_{\mathbb{L}^2} &= \left\langle |y|^2 M_2, \Delta^2 M_2\right\rangle_{\mathbb{L}^2}\\
&= \int_{\mathbb{R}^d}|y|^2|\Delta M_2|^2\,dx + 2\left\langle(y\cdot\Delta y + |\nabla y|^2)M_2 + 2(y\cdot\nabla y)\nabla M_2, \Delta M_2\right\rangle_{\mathbb{L}^2}\\
&\leq 2\left(|y|_{\mathbb{L}^\infty}|\Delta y|_{\mathbb{L}^2} + |\nabla y|^2_{\mathbb{L}^4} + 2|y|_{\mathbb{L}^\infty}|\nabla y|_{\mathbb{L}^4}|\nabla M_2|_{\mathbb{L}^4}\right)|\Delta M_2|_{\mathbb{L}^2} + |y|^2_{\mathbb{L}^\infty}|\Delta M_2|^2_{\mathbb{L}^2}\\
&\lesssim |y|^{1-\frac{d}{4}}_{\mathbb{L}^2}|\Delta y|^{1+\frac{d}{4}}_{\mathbb{L}^2}|\Delta M_2|_{\mathbb{L}^2} + \left(|y|^{2-\frac{d}{2}}_{\mathbb{L}^2}|\Delta y|^{\frac{d}{2}}_{\mathbb{L}^2} + |y|^{\frac{3}{2}-\frac{3d}{8}}_{\mathbb{L}^2}|\Delta y|^{\frac{1}{2}+\frac{3d}{8}}_{\mathbb{L}^2}\right)|\nabla M_2|_{\mathbb{H}^1}|\Delta M_2|_{\mathbb{L}^2}\\
&\lesssim |y|^2_{\mathbb{L}^2}\left(|\Delta M_2|^{\frac{8}{4-d}}_{\mathbb{L}^2} + |\nabla M_2|^{\frac{32}{12-3d}}_{\mathbb{H}^1}\right) + \delta|\Delta y|^2_{\mathbb{L}^2}.
\end{aligned}$$

For the $\Delta^2 y$ term,

$$\left\langle y, \left\langle M_2, \Delta^2 y\right\rangle M_1\right\rangle_{\mathbb{L}^2} = \left\langle (y\cdot M_1)M_2, \Delta^2 y\right\rangle_{\mathbb{L}^2}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^d} \langle \Delta y, M_1 \rangle \langle \Delta y, M_2 \rangle \,\mathrm{d}x \\
&\quad + \langle (y\cdot \Delta M_1) M_2 + (y\cdot M_1)\Delta M_2 + 2(y\cdot \nabla M_1)\nabla M_2, \Delta y\rangle_{\mathbb{L}^2} \\
&\quad + 2\langle (\nabla y \cdot \nabla M_1) M_2 + (\nabla y\cdot M_1)\nabla M_2, \Delta y\rangle_{\mathbb{L}^2} \\
&\le \int_{\mathbb{R}^d} \langle \Delta y, M_1 \rangle \langle \Delta y, M_2 \rangle \,\mathrm{d}x \\
&\quad + |y|_{\mathbb{L}^\infty} |\Delta y|_{\mathbb{L}^2} \left(|\Delta M_1|_{\mathbb{L}^2} + |\Delta M_2|_{\mathbb{L}^2} + |\nabla M_1|_{\mathbb{L}^4} |\nabla M_2|_{\mathbb{L}^4} \right) \\
&\quad + 2|\nabla y|_{\mathbb{L}^4} |\Delta y|_{\mathbb{L}^2} \left(|\nabla M_1|_{\mathbb{L}^4} + |\nabla M_2|_{\mathbb{L}^4} \right) \\
&\le \int_{\mathbb{R}^d} \langle \Delta y, M_1 \rangle \langle \Delta y, M_2 \rangle \,\mathrm{d}x \\
&\quad + c|y|_{\mathbb{L}^2}^{1-\frac{d}{4}} |\Delta y|_{\mathbb{L}^2}^{1+\frac{d}{4}} \left(|\Delta M_1|_{\mathbb{L}^2} + |\Delta M_2|_{\mathbb{L}^2} + |\nabla M_1|_{\mathbb{H}^1} |\nabla M_2|_{\mathbb{H}^1} \right) \\
&\quad + c|y|_{\mathbb{L}^2}^{\frac{1}{2}-\frac{d}{8}} |\Delta y|_{\mathbb{L}^2}^{\frac{3}{2}+\frac{d}{8}} \left(|\nabla M_1|_{\mathbb{H}^1} + |\nabla M_2|_{\mathbb{H}^1} \right) \\
&\le \int_{\mathbb{R}^d} \langle \Delta y, M_1 \rangle \langle \Delta y, M_2 \rangle \,\mathrm{d}x + c\delta |\Delta y|_{\mathbb{L}^2}^2 \\
&\quad + c(\delta^{-1}) \left(|\Delta M_1|_{\mathbb{L}^2} + |\Delta M_2|_{\mathbb{L}^2} + |\nabla M_1|_{\mathbb{H}^1} |\nabla M_2|_{\mathbb{H}^1} \right)^{\frac{8}{4-d}} |y|_{\mathbb{L}^2}^2 \\
&\quad + c(\delta^{-1}) \left(|\nabla M_1|_{\mathbb{H}^1} + |\nabla M_2|_{\mathbb{H}^1} \right)^{\frac{16}{4-d}} |y|_{\mathbb{L}^2}^2,
\end{aligned}
$$

where

$$
\begin{aligned}
\int_{\mathbb{R}^d} \langle \Delta y, M_1 \rangle \langle \Delta y, M_2 \rangle \,\mathrm{d}x &= \frac{1}{2} \int_{\mathbb{R}^d} \langle \Delta y, M_1 \rangle \left(|\nabla M_2|^2 + \langle M_2, \Delta M_1 \rangle \right) \,\mathrm{d}s \\
&\quad - \frac{1}{2} \int_{\mathbb{R}^d} \langle \Delta y, M_2 \rangle \left(|\nabla M_1|^2 + \langle M_1, \Delta M_2 \rangle \right) \,\mathrm{d}x \\
&= \frac{1}{2} \int_{\mathbb{R}^d} \langle \Delta y, y \rangle \left(|\nabla M_2|^2 + \langle M_2, \Delta M_1 \rangle \right) \,\mathrm{d}x \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^d} \langle \Delta y, M_2 \rangle \left(\langle \nabla (M_1 + M_2), -\nabla y \rangle + \langle M_2, \Delta y \rangle - \langle y, \Delta M_2 \rangle \right) \,\mathrm{d}x \\
&\le \frac{1}{2} |\Delta y|_{\mathbb{L}^2} |y|_{\mathbb{L}^\infty} \left(|\nabla M_2|_{\mathbb{L}^4}^2 + |\Delta M_1|_{\mathbb{L}^2} + |\Delta M_2|_{\mathbb{L}^2} \right) \\
&\quad + \frac{1}{2} |\Delta y|_{\mathbb{L}^2} |\nabla y|_{\mathbb{L}^4} |\nabla (M_1 + M_2)|_{\mathbb{L}^4} + \frac{1}{2} |\Delta y|_{\mathbb{L}^2}^2 \\
&\le \frac{1}{2} |\Delta y|_{\mathbb{L}^2}^2 + c|y|_{\mathbb{L}^2}^{1-\frac{d}{4}} |\Delta y|_{\mathbb{L}^2}^{1+\frac{d}{4}} \left(|\nabla M_2|_{\mathbb{H}^1}^2 + |\Delta M_1|_{\mathbb{L}^2} + |\Delta M_2|_{\mathbb{L}^2} \right) \\
&\quad + c|y|_{\mathbb{L}^2}^{\frac{1}{2}-\frac{d}{8}} |\Delta y|_{\mathbb{L}^2}^{\frac{3}{2}+\frac{d}{8}} |\nabla (M_1 + M_2)|_{\mathbb{H}^1} \\
&\le \frac{1}{2} |\Delta y|_{\mathbb{L}^2}^2 + c\delta |\Delta y|_{\mathbb{L}^2}^2 \\
&\quad + c(\delta^{-1}) \left(\left(|\nabla M_2|_{\mathbb{H}^1}^2 + |\Delta M_1|_{\mathbb{L}^2} + |\Delta M_2|_{\mathbb{L}^2} \right)^{\frac{8}{4-d}} + |\nabla (M_1 + M_2)|_{\mathbb{H}^1}^{\frac{16}{4-d}} \right) |y|_{\mathbb{L}^2}^2.
\end{aligned}
$$

Thus,

$$
\frac{1}{2} \mathrm{d}|y(t)|_{\mathbb{L}^2}^2 \le \phi(t) |y(t)|_{\mathbb{L}^2}^2 \,\mathrm{d}t + \alpha \left(c_1 \delta - \frac{1}{2} \right) |\Delta y(t)|_{\mathbb{L}^2}^2 \,\mathrm{d}t - \sum_k \left\langle y, \nabla_{g_k} y \right\rangle_{\mathbb{L}^2} \,\mathrm{d}W_k,
$$

for some constant c_1 and

$$
\phi(t) = c(h, \delta^{-1}) \left(1 + |\nabla M_1|_{\mathbb{H}^1}^q + |\nabla M_2|_{\mathbb{H}^1}^q \right),
$$

for some $q \in [2, \infty)$ depending on d. Since $\nabla M_1, \nabla M_2$ in $L^{2p}(\Omega; L^\infty(0,T;\mathbb{H}^1))$, we have $\int_0^T \phi(t)\,\mathrm{d}t < \infty$, $\mathbb{P}$-a.s. Let δ be sufficiently small such that $c_1\delta - \frac{1}{2} < 0$.

If $\mathrm{div} g_k = 0$ for all k, then applying Gronwall's inequality directly and using the fact $y(0) = 0$,

$$|y(t)|^2_{\mathbb{L}^2} \le |y(0)|^2_{\mathbb{L}^2} e^{\int_0^T \phi(t)\ \mathrm{d}t} = 0, \quad \mathbb{P}\text{-a.s.}$$

Otherwise, let $X(t) := \frac{1}{2}|y(t)|^2_{\mathbb{L}^2} e^{-2\int_0^t \phi(s)\ \mathrm{d}s}$. Then we have

$$\begin{aligned} \mathrm{d}X(t) &= \frac{1}{2}\left\langle \mathrm{d}|y(t)|^2_{\mathbb{L}^2}, e^{-2\int_0^t \phi(s)\ \mathrm{d}s}\right\rangle + \frac{1}{2}\left\langle |y(t)|^2_{\mathbb{L}^2}, -2\phi(t) e^{-2\int_0^t \phi(s)\ \mathrm{d}s}\right\rangle \\ &\quad + \frac{1}{2}\left\langle \mathrm{d}|y(t)|^2_{\mathbb{L}^2}, \mathrm{d}e^{-2\int_0^t \phi(s)\ \mathrm{d}s}\right\rangle \\ &\le -e^{-2\int_0^t \phi(s)\ \mathrm{d}s} \sum_k \left\langle y, \nabla_{g_k} y\right\rangle\ \mathrm{d}W_k, \end{aligned}$$

where the process $M(t) := \sum_k \int_0^t e^{-2\int_0^t \phi(s)\ \mathrm{d}s} \langle y, \nabla_{g_k} y\ \mathrm{d}W_k\rangle_{\mathbb{L}^2}$ is a martingale. Hence, $\mathbb{E}[X(t)] \le \mathbb{E}[M(t)] = 0$, implying that

$$\mathbb{E}\left[|y(t)|^2_{\mathbb{L}^2}\right] = 0, \quad t \in [0,T].$$

This proves the pathwise uniqueness of the solution of (2.3), concluding the proof of Theorem 2.1.

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APPENDIX A. PRELIMINARY ESTIMATES

A.1. **Useful formulae.** Let $u,w\in\mathbb{H}^2$ and $f,g\in H^1(\mathbb{R}^d;\mathbb{R}^d)$, $d=2,3$. Then

$$\begin{aligned}
(\nabla_g)(u\times w) &= \nabla_g u\times w + u\times\nabla_g w,\\
(\nabla_g)^2 u &= \sum_{i=1}^d g_i^2\partial_i^2 u + \sum_{i,j=1,i\neq j}^d g_i g_j \partial_{ij}u + \nabla_{\nabla_g g}u,\\
\nabla_f(\nabla_g u) &= \sum_{i,j=1}^d f_i g_j\partial_{ij}u + \nabla_{\nabla_f g}u.
\end{aligned}$$

For u,w with suitable decay properties,

$$\left\langle \nabla_f u, w\right\rangle_{\mathbb{L}^2} = -\left\langle u, \nabla_f w + (\operatorname{div} f)w\right\rangle_{\mathbb{L}^2}, \tag{A.1}$$

$$\left\langle \nabla_f u, \nabla_g w\right\rangle_{\mathbb{L}^2} = -\sum_{i,j=1}^d \left\langle f_i g_j \partial_{ij}u + \partial_j(f_i g_j)\partial_i u, w\right\rangle_{\mathbb{L}^2}. \tag{A.2}$$

By (A.1), for $f\in W^{2,\infty}(\mathbb{R}^d;\mathbb{R}^d)$,

$$\left\langle u, \nabla_f u\right\rangle_{\mathbb{L}^2} = -\frac{1}{2}\left\langle (\operatorname{div} f)u, u\right\rangle_{\mathbb{L}^2}, \tag{A.3}$$

and

$$\begin{aligned}
-\left\langle |u|^2 u, \nabla_f u\right\rangle_{\mathbb{L}^2} &= \left\langle \nabla_f(|u|^2 u) + (\operatorname{div} f)|u|^2 u, u\right\rangle_{\mathbb{L}^2}\\
&= 3\left\langle |u|^2\nabla_f u, u\right\rangle_{\mathbb{L}^2} + \left\langle (\operatorname{div} f)|u|^2 u, u\right\rangle_{\mathbb{L}^2}\\
&= \frac{1}{4}\left\langle (\operatorname{div} f)|u|^2 u, u\right\rangle_{\mathbb{L}^2},
\end{aligned}$$

which imply

$$\begin{aligned}
\left\langle (1-|u|^2)u, \nabla_f u\right\rangle_{\mathbb{L}^2} &= \frac{1}{4}\int_{\mathbb{R}^d} (\operatorname{div} f)\left(|u|^4 - 2|u|^2\right)\,\mathrm{d}x\\
&= \frac{1}{4}\int_{\mathbb{R}^d} (\operatorname{div} f)(1-|u|^2)^2\,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^d}\operatorname{div} f\,\mathrm{d}x,
\end{aligned} \tag{A.4}$$

where $\int_{\mathbb{R}^d}\operatorname{div} f\,\mathrm{d}x = 0$ if f vanishes at infinity.

Without loss of generality, assume that j is supported on the (open) unit ball. Let $B_\varepsilon(x)$ denote the open ball of radius $\varepsilon>0$ centred at $x\in\mathbb{R}^d$. Then for weighted $\mathbb{L}^2$-norm,

$$\begin{aligned}
|J_\varepsilon u|^2_{\mathbb{L}^2_\rho} &= \int_{\mathbb{R}^d}\left|\int_{B_\varepsilon(x)} j_\varepsilon(x-y)u(y)\,\mathrm{d}y\right|^2\rho(x)\,\mathrm{d}x\\
&\le \int_{\mathbb{R}^d}\left(\int_{B_\varepsilon(x)} j_\varepsilon(x-y)\rho^{-1}(y)\,\mathrm{d}y\right)\left(\int_{\mathbb{R}^d} j_\varepsilon(x-y)|u(y)|^2\rho(y)\,\mathrm{d}y\right)\rho(x)\,\mathrm{d}x,
\end{aligned}$$

where $\rho^{-1}(y) = (1+|y|^2)^2$ is bounded from above by $(1+(|x|+\varepsilon)^2)^2$ in $B_\varepsilon(x)$, implying

$$\begin{aligned}
\int_{B_\varepsilon(x)} j_\varepsilon(x-y)\rho^{-1}(y)\rho(x)\,\mathrm{d}y &\le (1+(|x|+\varepsilon)^2)^2\rho(x)\int_{B_\varepsilon(x)} j_\varepsilon(x-y)\,\mathrm{d}y\\
&= (1+(|x|+\varepsilon)^2)^2\rho(x)\\
&\le c, \quad \forall x\in\mathbb{R}^d.
\end{aligned}$$

Thus,

$$\begin{aligned}
|J_\varepsilon u|^2_{\mathbb{L}^2_\rho} &\le c\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} j_\varepsilon(x-y)|u(y)|^2\rho(y)\,\mathrm{d}y\,\mathrm{d}x\\
&= c\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} j_\varepsilon(x-y)\,\mathrm{d}x\right)|u(y)|^2\rho(y)\,\mathrm{d}y\\
&= c|u|^2_{\mathbb{L}^2_\rho}.
\end{aligned} \tag{A.5}$$

Similarly,

$$\begin{aligned}\langle J_\varepsilon u, w\rangle_{\mathbb{L}^2_\rho} &= \langle u, J_\varepsilon(w\rho)\rangle_{\mathbb{L}^2}\\ &= \langle u, J_\varepsilon w\rangle_{\mathbb{L}^2_\rho} + \langle u, J_\varepsilon(w\rho) - w\rho\rangle_{\mathbb{L}^2} + \langle u, w - J_\varepsilon w\rangle_{\mathbb{L}^2_\rho}\\ &\le \langle u, J_\varepsilon w\rangle_{\mathbb{L}^2_\rho} + |u|_{\mathbb{L}^2}|J_\varepsilon(w\rho) - w\rho|_{\mathbb{L}^2} + |u|_{\mathbb{L}^2_\rho}|J_\varepsilon w - w|_{\mathbb{L}^2},\end{aligned} \tag{A.6}$$

where the last two terms converge to 0 as $\varepsilon \to 0$ for $u, w \in \mathbb{L}^2$.

A.2. **Estimates for mixed partial derivatives.**

Lemma A.1. *Let $u : \mathbb{R}^d \to \mathbb{R}^3$ be a sufficiently smooth function with suitable decay properties. Then*

$$\sum_{i=1}^d |\partial_i^2 u|_{\mathbb{L}^2}^2 \le |\Delta u|_{\mathbb{L}^2}^2, \quad \sum_{i,j=1}^d |\partial_{ij} u|_{\mathbb{L}^2}^2 \le |\Delta u|_{\mathbb{L}^2}^2. \tag{A.7}$$

Let $f \in W^{2,\infty}(\mathbb{R}^d;\mathbb{R})$. Then for $i, j \in \{1,\dots,d\}$,

$$\left\langle \partial_i^4 u, f\partial_j u\right\rangle_{\mathbb{L}^2} \lesssim |\partial_j u|_{\mathbb{L}^2}^2 + |\partial_i^2 u|_{\mathbb{L}^2}^2 + |\partial_{ij} u|_{\mathbb{L}^2}^2, \tag{A.8}$$

$$\left\langle \partial_i^2\partial_j^2 u, f\partial_i u\right\rangle_{\mathbb{L}^2} \lesssim |\partial_i u|_{\mathbb{L}^2}^2 + |\partial_i^2 u|_{\mathbb{L}^2}^2 + |\partial_j^2 u|_{\mathbb{L}^2}^2 + |\partial_{ij} u|_{\mathbb{L}^2}^2, \tag{A.9}$$

$$\left\langle \partial_i^3\partial_j u, f\partial_i u\right\rangle_{\mathbb{L}^2} \lesssim |\partial_i u|_{\mathbb{L}^2}^2 + |\partial_i^2 u|_{\mathbb{L}^2}^2 + |\partial_{ij} u|_{\mathbb{L}^2}^2, \tag{A.10}$$

$$\left\langle \partial_i^3\partial_j u, f\partial_j u\right\rangle_{\mathbb{L}^2} \lesssim |\partial_j u|_{\mathbb{L}^2}^2 + |\partial_i^2 u|_{\mathbb{L}^2}^2 + |\partial_j^2 u|_{\mathbb{L}^2}^2 + |\partial_{ij} u|_{\mathbb{L}^2}^2. \tag{A.11}$$

Proof. For the inequalities in (A.7), we first observe that

$$|\partial_{ij} u|_{\mathbb{L}^2}^2 = \left\langle \partial_i^2 u, \partial_j^2 u\right\rangle_{\mathbb{L}^2}.$$

Then we have

$$\sum_{i=1}^d |\partial_i^2 u|_{\mathbb{L}^2}^2 = |\Delta u|_{\mathbb{L}^2}^2 - 2\sum_{i,j=1, i\neq j}^d \left\langle \partial_i^2 u, \partial_j^2 u\right\rangle_{\mathbb{L}^2} = |\Delta u|_{\mathbb{L}^2}^2 - 2\sum_{i,j=1, i\neq j}^d |\partial_{ij} u|_{\mathbb{L}^2}^2 \le |\Delta u|_{\mathbb{L}^2}^2,$$

and as a result,

$$\sum_{i,j=1}^d |\partial_{ij} u|_{\mathbb{L}^2}^2 \le \frac{1}{2}\left(\sum_{i=1}^d |\partial_i^2 u|_{\mathbb{L}^2}^2 + \sum_{j=1}^d |\partial_j^2 u|_{\mathbb{L}^2}^2\right) \le |\Delta u|_{\mathbb{L}^2}^2.$$

In order to deduce the inequalities (A.8)-(A.11), we first observe that

$$\left\langle \partial_i^2 u, f\partial_i^2\partial_j u\right\rangle_{\mathbb{L}^2} = -\left\langle \partial_i^2 u, \partial_j f\, \partial_i^2 u + f\partial_i^2\partial_j u\right\rangle_{\mathbb{L}^2} = -\frac{1}{2}\left\langle \partial_i^2 u, \partial_j f\, \partial_i^2 u\right\rangle_{\mathbb{L}^2}. \tag{A.12}$$

Then for (A.8),

$$\begin{aligned}\left\langle \partial_i^4 u, f\partial_j u\right\rangle_{\mathbb{L}^2} &= \left\langle \partial_i^2 u, \partial_i^2 f\, \partial_j u + f\partial_i^2\partial_j u + 2\partial_i f\, \partial_{ij} u\right\rangle_{\mathbb{L}^2}\\ &= \left\langle \partial_i^2 u, \partial_i^2 f\, \partial_j u + 2\partial_i f\, \partial_{ij} u\right\rangle_{\mathbb{L}^2} - \frac{1}{2}\left\langle \partial_i^2 u, \partial_j f\, \partial_i^2 u\right\rangle_{\mathbb{L}^2}\\ &\lesssim |\partial_j u|_{\mathbb{L}^2}^2 + |\partial_i^2 u|_{\mathbb{L}^2}^2 + |\partial_{ij} u|_{\mathbb{L}^2}^2,\end{aligned}$$

where the second equality holds by (A.12). Similarly, for (A.9),

$$\begin{aligned}\left\langle \partial_i^2\partial_j^2 u, f\partial_i u\right\rangle_{\mathbb{L}^2} &= \left\langle \partial_j^2 u, \partial_i^2 f\, \partial_i u + f\partial_i^3 u + 2\partial_i f\, \partial_i^2 u\right\rangle_{\mathbb{L}^2}\\ &= \left\langle \partial_j^2 u, \partial_i^2 f\, \partial_i u + 2\partial_i f\, \partial_i^2 u\right\rangle_{\mathbb{L}^2} - \left\langle \partial_i f\, \partial_j^2 u, \partial_i^2 u\right\rangle_{\mathbb{L}^2} - \left\langle f\, \partial_i\partial_j^2 u, \partial_i^2 u\right\rangle_{\mathbb{L}^2}\\ &= \left\langle \partial_j^2 u, \partial_i^2 f\, \partial_i u + \partial_i f\, \partial_i^2 u\right\rangle_{\mathbb{L}^2} - \left\langle \partial_i u, \partial_j^2\left(f\partial_i^2 u\right)\right\rangle_{\mathbb{L}^2}\\ &= \left\langle \partial_j^2 u, \partial_i^2 f\, \partial_i u + \partial_i f\, \partial_i^2 u\right\rangle_{\mathbb{L}^2} - \left\langle \partial_i u, \partial_j^2 f\, \partial_i^2 u\right\rangle_{\mathbb{L}^2} - 2\left\langle \partial_i u, \partial_j f\, \partial_j\partial_i^2 u\right\rangle_{\mathbb{L}^2}\\ &\quad - \left\langle f\partial_i u, \partial_i^2\partial_j^2 u\right\rangle_{\mathbb{L}^2}\\ &= \frac{1}{2}\left\langle \partial_j^2 u, \partial_i^2 f\, \partial_i u + \partial_i f\, \partial_i^2 u\right\rangle_{\mathbb{L}^2} - \frac{1}{2}\left\langle \partial_i u, \partial_j^2 f\, \partial_i^2 u\right\rangle_{\mathbb{L}^2} - \left\langle \partial_i u, \partial_j f\, \partial_j\partial_i^2 u\right\rangle_{\mathbb{L}^2}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left\langle \partial_i u, \partial_i^2 f\, \partial_j^2 u - \partial_j^2 f\, \partial_i^2 u\right\rangle_{\mathbb{L}^2} + \frac{1}{2}\left\langle \partial_j^2 u, \partial_i f\, \partial_i^2 u\right\rangle_{\mathbb{L}^2} + \left\langle \partial_j^2 f\, \partial_i u + \partial_j f\, \partial_{ij} u, \partial_i^2 u\right\rangle_{\mathbb{L}^2} \\
&\lesssim |\partial_i u|_{\mathbb{L}^2}^2 + |\partial_i^2 u|_{\mathbb{L}^2}^2 + |\partial_j^2 u|_{\mathbb{L}^2}^2 + |\partial_{ij} u|_{\mathbb{L}^2}^2,
\end{aligned}
$$

where the fifth equality holds since the last term in the fourth equality is the negative counterpart of the left-hand side. For (A.10),

$$
\begin{aligned}
\left\langle \partial_i^3 \partial_j u, f \partial_i u\right\rangle_{\mathbb{L}^2} &= \left\langle \partial_{ij} u, \partial_i^2 (f \partial_i u)\right\rangle_{\mathbb{L}^2} \\
&= \left\langle \partial_{ij} u, \partial_i^2 f\, \partial_i u + f\, \partial_i^3 u + 2\partial_i f\, \partial_i^2 u\right\rangle_{\mathbb{L}^2} \\
&= \left\langle \partial_{ij} u, \partial_i^2 f\, \partial_i u + 2\partial_i f\, \partial_i^2 u\right\rangle_{\mathbb{L}^2} - \left\langle \partial_i u, \partial_j f\, \partial_i^3 u\right\rangle_{\mathbb{L}^2} - \left\langle \partial_i u, f \partial_i^3 \partial_j u\right\rangle_{\mathbb{L}^2} \\
&= \frac{1}{2}\left\langle \partial_{ij} u, \partial_i^2 f\, \partial_i u + 2\partial_i f\, \partial_i^2 u\right\rangle_{\mathbb{L}^2} - \frac{1}{2}\left\langle \partial_i u, \partial_j f\, \partial_i^3 u\right\rangle_{\mathbb{L}^2} \\
&= \frac{1}{2}\left\langle \partial_{ij} u, \partial_i^2 f\, \partial_i u\right\rangle_{\mathbb{L}^2} + \left\langle \partial_{ij} u, \partial_i f\, \partial_i^2 u\right\rangle_{\mathbb{L}^2} + \frac{1}{2}\left\langle \partial_j f\, \partial_i^2 u + \partial_{ij} f\, \partial_i u, \partial_i^2 u\right\rangle_{\mathbb{L}^2} \\
&\lesssim |\partial_i u|_{\mathbb{L}^2}^2 + |\partial_i^2 u|_{\mathbb{L}^2}^2 + |\partial_{ij} u|_{\mathbb{L}^2}^2.
\end{aligned}
$$

For (A.11), the derivation is similar to that of (A.10) and we obtain

$$
\begin{aligned}
\left\langle \partial_i^3 \partial_j u, f \partial_j u\right\rangle_{\mathbb{L}^2} &= \frac{1}{2}\left\langle \partial_{ij} u, \partial_i^2 f\, \partial_j u\right\rangle_{\mathbb{L}^2} + \left\langle \partial_{ij} u, \partial_i f\, \partial_{ij} u\right\rangle_{\mathbb{L}^2} + \frac{1}{2}\left\langle \partial_i f\, \partial_j^2 u + \partial_{ij} f\, \partial_j u, \partial_i^2 u\right\rangle_{\mathbb{L}^2} \\
&\lesssim |\partial_j u|_{\mathbb{L}^2}^2 + |\partial_i^2 u|_{\mathbb{L}^2}^2 + |\partial_j^2 u|^2 + |\partial_{ij} u|_{\mathbb{L}^2}^2,
\end{aligned}
$$

concluding the proof. □

This lemma allows us to treat a higher-order term $\langle \Delta^2(\nabla_f u), \nabla_f u\rangle$. Note that

$$
\begin{aligned}
\Delta^2(\nabla_f u) &= \nabla_f(\Delta^2 u) + 4\sum_{i,j=1}^{d} \partial_i f_j \partial_{ij}(\Delta u) + 2\left(\nabla_{\Delta f}(\Delta u) + 2\sum_{i,j,k=1}^{d} \partial_{ij} f_k \partial_{ijk} u\right) \\
&\quad + \left(4\sum_{i,j=1}^{d} \partial_i(\Delta f_j)\partial_{ij} u + \nabla_{\Delta^2 f} u\right) \\
&= \nabla_f(\Delta^2 u) + 4\mathrm{T}_{1a}(u) + 2\mathrm{T}_{1b}(u) + \mathrm{T}_{1c}(u).
\end{aligned}
$$

For $f \in H^4(\mathbb{R}^d;\mathbb{R}^d)$, using Lemma A.1 we have

$$
\begin{aligned}
\left\langle \mathrm{T}_{1a}(u), \nabla_f u\right\rangle_{\mathbb{L}^2} &= \sum_{i,j} \left\langle \partial_i f_j \partial_{ij}(\Delta u), \nabla_f u\right\rangle_{\mathbb{L}^2} \\
&\lesssim |f|_{W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)}^2 |\nabla u|_{\mathbb{H}^1}^2, \\
\left\langle \mathrm{T}_{1b}(u), \nabla_f u\right\rangle_{\mathbb{L}^2} &= \left\langle \nabla_{\Delta f}(\Delta u), \nabla_f u\right\rangle_{\mathbb{L}^2} + 2\sum_{i,j,k} \left\langle \partial_{ij} f_k\, \partial_{ijk} u, \nabla_f u\right\rangle_{\mathbb{L}^2} \\
&= -\left\langle \Delta u, \nabla_{\Delta f}\nabla_f u + (\mathrm{div}\Delta f)\nabla_f u\right\rangle_{\mathbb{L}^2} - 2\sum_{i,j,k} \left\langle \partial_{ij} u, \partial_k\left(\partial_{ij} f_k \nabla_f u\right)\right\rangle_{\mathbb{L}^2} \\
&\lesssim |f|_{W^{2,\infty}\cap W^{3,4}(\mathbb{R}^d;\mathbb{R}^d)}^2 |\nabla u|_{\mathbb{H}^1}^2, \\
\left\langle \mathrm{T}_{1c}(u), \nabla_f u\right\rangle_{\mathbb{L}^2} &= 4\sum_{i,j} \left\langle \partial_i(\Delta f_j)\partial_{ij} u, \nabla_f u\right\rangle_{\mathbb{L}^2} + \left\langle \nabla_{\Delta^2 f} u, \nabla_f u\right\rangle_{\mathbb{L}^2} \\
&\lesssim |f|_{W^{2,\infty}\cap W^{3,4}\cap H^4(\mathbb{R}^d;\mathbb{R}^d)}^2 |\nabla u|_{\mathbb{H}^1}^2, \\
\left\langle \Delta^2 u, \nabla_f u\right\rangle_{\mathbb{L}^2} &\lesssim |f|_{W^{2,\infty}(\mathbb{R}^d;\mathbb{R}^d)}^2 |\nabla u|_{\mathbb{H}^1}^2,
\end{aligned} \tag{A.13}
$$

where $|f|_{W^{2,\infty}(\mathbb{R}^d;\mathbb{R}^d)} + |f|_{W^{3,4}(\mathbb{R}^d;\mathbb{R}^d)} \lesssim |f|_{H^4(\mathbb{R}^d;\mathbb{R}^d)}$ for $d \leq 3$. The estimates (A.13) help us to address Itô corrections in Step 2(ii) of the proof of Lemma 3.2.

APPENDIX B. ON THE TORUS

For equation (2.3) formulated on the torus $\mathbb{T}^d \subset \mathbb{R}^d$, Theorem 2.1 still holds with $\mathbb{T}^d$ in place of $\mathbb{R}^d$. Since the domain is bounded, the proof can be simplified in this case. In this section, let $\mathbb{L}^p$ and $\mathbb{H}^\sigma$ denote the usual Lebesgue and Hilbert spaces $L^p(\mathbb{T}^d;\mathbb{R}^3)$ and $W^{\sigma,2}(\mathbb{T}^d;\mathbb{R}^3)$, respectively. We can work with M directly and only need Faedo-Galerkin approximations $\{M_n\}$ with a cut-off function ψ_R that depends on $|M(t)|_{\mathbb{L}^\infty}$ instead of $|M(t,x)|^2$. More explicitly, let $\{\mathbf{e}_i\}$ be an orthonormal basis of $\mathbb{L}^2$ consisting of eigenvectors of the negative Laplacian $-\Delta$ in $\mathbb{T}^d$ and let $\mathbb{H}_n := \operatorname{span}\{\mathbf{e}_i, \dots, \mathbf{e}_n\}$. Let Π_n be the orthogonal projection from $\mathbb{L}^2$ onto $\mathbb{H}_n$, and we formulate the approximating equation:

$$\mathrm{d}M_n(t) = F_n^R(M_n(t))\,\mathrm{d}t + \frac{1}{2}\sum_{k=1}^{\infty} S_{k,n}(M_n(t))\,\mathrm{d}t + \sum_{k=1}^{\infty} G_{k,n}(M_n(t))\,\mathrm{d}W_k(t), \quad m_n(0) = \Pi_n m_0, \tag{B.1}$$

where

$$\begin{aligned} F_n^R &: \mathbb{H}_n \ni u \mapsto \psi_R(|u|_{\mathbb{L}^\infty})\Pi_n \bar{F}(u) - \Pi_n \nabla_\nu u \in \mathbb{H}_n, \\ S_{k,n} &: \mathbb{H}_n \ni u \mapsto \Pi_n S_k(u) \in \mathbb{H}_n, \\ G_{k,n} &: \mathbb{H}_n \ni u \mapsto \Pi_n G_k(u) \in \mathbb{H}_n, \end{aligned}$$

are locally Lipschitz. The uniform estimates can be obtained as in Lemma 3.2, and tightness and compact embeddings hold for the usual Lebesgue and Sobolev spaces without weight or localisation. Then, for the new processes $\tilde{M}_n$ and $\tilde{M}$ (deduced from Skorohod theorem) with strong convergence in $L^{2p}(\tilde{\Omega}; L^2(0,T;\mathbb{L}^2))$, convergence of the cut-off function $\tilde{\psi}_n^R = \psi_R(|\tilde{M}_n|_{\mathbb{L}^\infty}) \to \tilde{\psi}^R = \psi_R(|\tilde{M}|_{\mathbb{L}^\infty})$ in $L^{2p}(\tilde{\Omega}; L^2(0,T))$ is immediate from Gagliardo-Nirenberg inequality:

$$\begin{aligned} \left|\tilde{\psi}_n^R(t) - \tilde{\psi}^R(t)\right| &\le \sup_{y\in\mathbb{R}} |\psi_R'(y)| \; \left||\tilde{M}_n(t)|_{\mathbb{L}^\infty} - |\tilde{M}(t)|_{\mathbb{L}^\infty}\right| \\ &\lesssim |\tilde{M}_n(t) - \tilde{M}(t)|_{\mathbb{L}^\infty} \\ &\lesssim |\tilde{M}_n(t) - \tilde{M}(t)|_{\mathbb{L}^2}^{1-\frac{d}{4}} |\tilde{M}_n(t) - \tilde{M}(t)|_{\mathbb{H}^2}^{\frac{d}{4}}. \end{aligned}$$

Thus, we obtain similar convergences to Lemma 4.1 in $\mathbb{L}^2 = L^2(\mathbb{T}^d;\mathbb{R}^3)$ without knowing $\tilde{\psi}^R(t) = 1$, and then we can deduce the existence of weak martingale solution to

$$\tilde{M}(t) = \tilde{M}_0 + \int_0^t \left(\tilde{\psi}^R \bar{F}(\tilde{M}) - \nabla_\nu \tilde{M}\right)(s)\,\mathrm{d}s + \frac{1}{2}\sum_k \int_0^t S_k(\tilde{M})(s)\,\mathrm{d}s + \sum_k \int_0^t G_k(\tilde{M})(s)\,\mathrm{d}\tilde{W}_k(s). \tag{B.2}$$

Finally, $|\tilde{M}(t,x)| = 1$ (and thus $\tilde{\psi}^R(t) = 1$ for $R > 1$) for a.e.-(t,x), $\tilde{\mathbb{P}}$-a.s. can be shown using a similar but simpler calculation to Lemma 3.3, and the pathwise uniqueness follows as in Section 5.2.

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