NON-SOLVABLE TORSION-FREE VIRTUALLY SOLVABLE GROUPS

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ABSTRACT. We show that a non-solvable, torsion-free, virtually solvable group S must have Hirsch length $h(S) \geqslant 10$. If $h(S) < 14$ then A_5 is the only simple factor. If S is virtually nilpotent and $h(S) \leqslant 14$ then is Fitting subgroup has nilpotency class $\leqslant 3$.

We shall consider here the question: *What is the smallest torsion-free virtually solvable group which is not solvable?* Here "smallest" should be interpreted as having minimal Hirsch length. Lutowski and Szczepański have recently shown that 15 is the smallest dimension in which there are non-solvable Bieberbach groups, and have given two explicit examples in this dimension, one with holonomy $A_5 = PSL(2,5)$ and another with holonomy $PSL(2,7)$ [7]. The question remains open for groups which are not virtually abelian. We shall show that if there is such a group S with Hirsch length $h(S) < 15$ then $h(S) \geqslant 10$ and the quotient of S by its solvable radical is A_5 or $PSL(2,7)$. If S is virtually nilpotent then its Fitting subgroup has nilpotency class $\leqslant 3$. Our arguments rest upon the groups in question having finite perfect quotients which act effectively on a free abelian group of small rank.

The first section is on notation and terminology, and §2 contains five lemmas. In §3 we define the notion of minimal TFNS group and show that such groups have crystallographic quotients. The next section uses knowledge of the finite subgroups of $GL(k,\mathbb{Z})$ and $Sp(2\ell,\mathbb{Z})$ for k and ℓ small [4, 8] to find the relevant minimal perfect groups and their representations. In §5 we use some commutative algebra to show that if a torsion-free, virtually solvable group S is neither solvable nor virtually nilpotent then $h(S) \geqslant 9$. In §6 we consider the commutator pairing for nilpotent groups, and we apply the work of §6 in §7 to minimal TFNS groups which are virtually nilpotent. Up to this point the condition that S be torsion-free is not prominent in our arguments. It is used in an essential way in §8, where we show that $h(S) \geqslant 10$,

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and limit the structure of the Fitting subgroup when S is virtually nilpotent. The final brief section contains a few questions.

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1. GENERALITIES

If G is a group then G', and ζG shall denote the commutator subgroup, and centre of G, respectively. If J is a subgroup of G then $C_G(J)$ is the centralizer of J in G. Let $G^{(0)} = G$ and $G^{(n+1)} = G^{(n)\prime}$ be the terms of the derived series for G, and let $\gamma_1 G = G$ and $\gamma_{n+1}G = [G, \gamma_n G]$ be the terms of the lower central series for G. (This notation is from [9].) Let $I(G)$ be the *isolator* of G' in G, so that $G/I(G)$ is the maximal torsion-free abelian quotient of G. Let $\sqrt{G}$ denote the Hirsch-Plotkin radical of G. If G is virtually polycyclic then $\sqrt{G}$ is the unique maximal nilpotent normal subgroup of G, and is also known as the Fitting subgroup of G. We shall use "simple" to mean "non-abelian and simple" throughout.

A virtually solvable group S has a solvable radical $\widetilde{S}$ of finite index. If S is not solvable then the lowest term of a composition series for $S/\widetilde{S}$ is a finite simple group, and so contains a minimal simple group [1]. We shall say that S is *TFNS* if it is torsion-free and not solvable, and is *minimal TFNS* if it is finitely generated, torsion-free, $S/\widetilde{S}$ is a minimal simple group and S has minimal Hirsch length for such groups. Thus $h(S) \leqslant 15$. (Cf. [7, Definition 2.1].)

An *outer action* of a group H on a group G is a homomorphism $\theta : H \to Out(G)$. Such a homomorphism induces homomorphisms from H to $Aut(G/G')$ and $Aut(\zeta G)$. We shall say that the action is *effective* if it is a monomorphism.

A *crystallographic* group G is a group which is an extension ξ of a finite group H by a finitely generated free abelian group A, such that the action $\theta : H \to Aut(A)$ induced by conjugation in G is effective. The *holonomy group* of G is the quotient G/A. Such extensions are classified by θ and a cohomology class $c(\xi)$ in $H^2(H; A^\theta)$, where H acts on A via θ. (We shall write just $H^2(H; A)$, when the action is clear.)

A *Bieberbach* group is a torsion-free crystallographic group. A crystallographic group corresponding to an extension ξ is a Bieberbach group if and only if $c(\xi)$ has non-zero restriction in $H^2(C; A)$ for every cyclic subgroup $C < H$ of prime order > 1 [12, Theorem 3.1].

2. SOME USEFUL LEMMAS

We have adapted the first two lemmas in this section for our needs.

Lemma 1. [12, Theorem 2.2] *Let G be a finitely generated, virtually abelian group. Then G is a crystallographic group if and only if it has no non-trivial finite normal subgroup. In that case $\sqrt{G}$ is free abelian and is the maximal normal abelian subgroup of G, and G has holonomy $H = G/\sqrt{G}$. If H is not solvable then $h(G) \geqslant 4$.*

Proof. We may assume that G is an extension of a finite group H by a finitely generated free abelian normal subgroup A. Then A has finite index in $\sqrt{G}$, and so $\sqrt{G}$ is nilpotent and has finite torsion subgroup.

Let $C = C_G(A)$ be the centralizer of A in G. Then G/C acts effectively on A. Since $[C : A]$ is finite, C' is finite, by a theorem of Schur [9, 10.1.4]. Hence if G has no non-trivial finite normal subgroup then $C' = 1$ and C is free abelian. By the same reasoning, $\sqrt{G}$ is abelian and so $C = \sqrt{G}$ is the maximal abelian normal subgroup of G. Since G/C acts effectively on A it acts effectively on C. Thus G is a crystallographic group with holonomy $H_1 = G/C$.

Conversely, suppose that G is a crystallographic group with holonomy H. if F is a finite normal subgroup of G then $A \cap F = 1$, since A is torsion-free, and so F projects injectively to H. Moreover, since A and F are each normal, $[A, F] \leqslant A \cap F = 1$ and so $AF \cong A \times F$. Since the action of H on A is effective we have $F = 1$.

The final observation holds since finite subgroups of $GL(3, \mathbb{Q})$ are solvable. □

In this context the Fitting subgroup $\sqrt{G}$ is also called the *translation* subgroup of G.

Lemma 2. [12, Prop. 4.1] *Let G be a crystallographic group with holonomy H. If $H^1(G; \mathbb{Q}) = 0$ then $Hom(\sqrt{G}, \mathbb{Q})^H = 1$. Hence the $\mathbb{Q}[G]$-module $\mathbb{Q} \otimes \sqrt{G}$ has no summand $\mathbb{Q}$ with trivial H-action.*

Proof. This follows from the exact sequence of low degree in the LHS spectral sequence for G as an extension of H by $\sqrt{G}$, since $H^1(G; \mathbb{Q}) = H^2(H; \mathbb{Q}) = 0$. The second assertion is then clear. □

Let N be a nilpotent group, and let tN be its torsion subgroup. Then N/tN is torsion-free and has a central series with torsion-free abelian subquotients [9, 5.2.7 and 5.2.20]. Let

$$\tilde{\gamma}_1 N = N > \cdots > \tilde{\gamma}_c N = tN$$

be the preimage in N of the most rapidly descending such central series. Then $\tilde{\gamma}_2 N = I(N)$. Let $\mathbb{Q}N^{ab} = \mathbb{Q} \otimes N/N' = \mathbb{Q} \otimes N/\tilde{\gamma}_2 N$.

Lemma 3. *Let G be a torsion-free virtually solvable group of finite Hirsch length and let $N = \sqrt{G}$. Suppose that G/N is a torsion group.*

Then the homomorphism from G/N to $Aut(\mathbb{Q}N^{ab})$ induced by conjugation in G is a monomorphism and G/N is finite. If G is finitely generated then $G/I(N)$ is crystallographic.

Proof. Since G/N is a torsion group $h(N) = h(G) < \infty$, and so $r = h(N^{ab})$ is finite. Let C be the kernel of the homomorphism from G to $Aut(\mathbb{Q}N^{ab}) \cong GL(r, \mathbb{Q})$ induced by conjugation in G. Then $N \leqslant C$ and $C/I(N)$ contains $N/I(N)$ as a central subgroup of finite index. Conjugation by elements of C also induces the identity on the subquotients $\tilde{\gamma}_i N/\tilde{\gamma}_{i+1}N$, for all $i > 1$. Hence C is nilpotent, by Baer's extension of a theorem of Schur [9, 14.5.1] and the fact that G is torsion-free. Hence $C = N$, by the maximality of N, and so G/N embeds in $Aut(\mathbb{Q}N^{ab})$. Since G/N is a torsion subgroup of $GL(r, \mathbb{Q})$ it is finite [9, 8.1.11].

This monomorphism factors through $Aut(N/I(N))$, and so G/N acts effectively on $N/I(N)$. If G is finitely generated then $N/I(N) \cong \mathbb{Z}^r$, and so $G/I(N)$ is a crystallographic group. □

Lemma 4. *Let S be a torsion-free solvable group such that $h(S)$ is finite. Then $\sqrt{S}$ is nilpotent, and $C_S(\sqrt{S}) = \zeta\sqrt{S}$.*

Proof. If N is a finitely generated subgroup of $\sqrt{S}$ then $h(N) \leqslant h = h(S)$, and so $\gamma_{h+1}N = 1$, since S is torsion-free. It follows immediately that $\gamma_{h+1}\sqrt{S} = 1$, and so $\sqrt{S}$ is nilpotent.

Let $C = C_S(\sqrt{S})$, and suppose that $\sqrt{S}$ is a proper subgroup of $C.\sqrt{S}$. Since S is solvable it has a subgroup D containing $C.\sqrt{S}$ and such that $D/\sqrt{S}$ is abelian. But then D is nilpotent, contradicting the maximality of the Hirsch-Plotkin radical. Hence $C = C_S(\sqrt{S}) \leqslant \sqrt{S}$ and so $C = \zeta\sqrt{S}$. □

In the virtually polycyclic case $S/\sqrt{S}$ is virtually abelian [9, 15.1.6]. However, this not so for all finitely presentable solvable groups [10].

If G is virtually polycyclic and all of its abelian subnormal subgroups have rank $\leqslant n$ then $G/\sqrt{G}$ is virtually abelian of rank $< n$ [15, Theorem 2]. Hence $h(G/\sqrt{G}) < h(\sqrt{G})$, since $\sqrt{G}$ contains all abelian subnormal subgroups. We may push this inequality a little further.

Lemma 5. *Let G be a virtually polycyclic group, and let V be a normal subgroup of finite index which contains $\sqrt{G}$ and such that $V/\sqrt{G}$ is abelian. Let C be the preimage in G of the centre of $V/\sqrt{G}'$. Then $h(G/\sqrt{G}) < h(\sqrt{G}/\sqrt{G}'.V' \cap C)$.*

Proof. We note first that $\sqrt{G} = \sqrt{V}$, since V is a normal subgroup and $\sqrt{G} \leqslant V$. Hence we may assume that $G = V$, since $h(G) = h(V)$, and so $G' \leqslant \sqrt{G}$. The preimage of $\sqrt{G/\sqrt{G}'}$ in G is $\sqrt{G}$ [9, 5.2.10], and so we may pass to the quotient $G/\sqrt{G}'$. Hence we may also assume that $\sqrt{G}$ is abelian. We then have $C = \zeta G \leqslant \sqrt{G}$ and $\sqrt{G/(G' \cap C)} = \sqrt{G}/(G' \cap C)$, since $G' \cap C$ is central. Applying [15, Theorem 2] to $G/G' \cap C$ gives the result. □

This bound is sharp. For example, if K is a totally real number field of degree 5, then the group of integral units $\mathcal{O}_K^\times$ has rank 4, and acts effectively on $\mathcal{O}_K$. Hence $G = \mathcal{O}_K \rtimes \mathcal{O}_K^\times$ is virtually torsion-free poly-$\mathbb{Z}$. The abelian normal subgroup $\mathcal{O}_K$ is its own centralizer in G, and so $\sqrt{G} = \mathcal{O}_K$ and $\zeta G = 1$. Hence $h(G/\sqrt{G}) = 4 < h(\sqrt{G}) = 5$.

3. minimal TFNS groups

If a virtually solvable group S is minimal TFNS then S^{ab} is finite. For otherwise there would be an epimorphism $\phi : S \to \mathbb{Z}$, and $\mathrm{Ker}(\phi)$ would not be solvable. Since $\mathrm{Ker}(\phi)$ is virtually solvable, it has finitely generated subgroups which are non-solvable, but have Hirsch length $< h(S)$, contradicting the minimality of S. We can improve on this.

Lemma 6. *Let S be a virtually solvable group such that $H = S/\widetilde{S}$ is perfect. If $\widetilde{S}^{(n)} = 1$ then $S^{(n)}$ is perfect, and if H is simple then $S^{(n)}/\widetilde{S^{(n)}} \cong H$.*

Proof. Let $f : S \to T = S/S^{(n+1)}$ be the natural epimorphism. Then f induces an epimorphism from H onto $T/f(\widetilde{S})$, since $f(\widetilde{S})$ is normal in T. Since H is perfect and T is solvable it follows that $f(\widetilde{S}) = T$, and so f induces epimorphisms $f^{(k)} : S^{(k)} \to T^{(k)}$, for all $k \geqslant 1$. Hence $T^{(n)} = 1$ and so $S^{(n)} = S^{(n+1)}$. Thus $S^{(n)}$ is perfect. The final assertion is clear, since H is the only non-solvable quotient of S. □

In particular, if S is minimal TFNS then it has a perfect subgroup of finite index which is also minimal TFNS. We may assume that S is finitely generated, since some finitely generated subgroup of S must map onto H. We may also assume that S is perfect, by Lemma 6.

Specializing further, we see that if G is a crystallographic group with perfect holonomy H then G' is a perfect crystallographic group, and if H is simple then G' also has holonomy H.

Lemma 7. *Let S be a finitely generated, infinite group which is torsion-free, virtually solvable and of finite Hirsch length. Then S has normal subgroups $V \leqslant U \leqslant \widetilde{S}$ such that S/U is crystallographic, U/V is finite, $V/I(V)$ has positive rank and U/V acts effectively on $V/I(V)$.*

Proof. Since S is finitely generated, infinite and virtually solvable it has a a normal subgroup $T \leqslant \widetilde{S}$ of finite index which is solvable and has infinite abelianization. The quotient S/T' is finitely generated and virtually abelian, and so has a maximal finite normal subgroup. Let U be the preimage in S of this subgroup. Then $A = \sqrt{S/U}$ has finite index in S/U and so is finitely generated. The torsion subgroup tA is trivial, since it is a finite normal subgroup of S/U. Since $A = \sqrt{S/U}$ and is torsion-free and virtually abelian, it is abelian. It is also its own centralizer in S/U, and so $(S/U)/A$ embeds in $Aut(A)$. Hence S/U is a crystallographic group.

Since $U \leqslant \widetilde{S}$ it is solvable. Let $U^{(i)}$ be the largest member of the derived series for U such that $U^{(i)}/I(U^{(i)})$ has positive rank, and let V be the preimage in U of the centralizer of $U^{(i)}/I(U^{(i)})$ in $U/I(U^{(i)})$. Then $U^{(i)} \leqslant V \leqslant U$, and so U/V is a torsion group. Since it acts effectively on $U^{(i)}/I(U^{(i)})$, which is a torsion-free abelian group of finite rank, it is finite [9, 8.1.11]. Since $U^{(i)}$ is central in V and $V/U^{(i)}$ is a torsion group, the commutator subgroup $V'I(U^{(i)})/I(U^{(i)})$ is also a torsion group. Thus $V/I(V)$ is torsion-free, of positive rank, and S/V is virtually abelian. □

If S is virtually nilpotent we may take $V = U = I(\sqrt{S})$, by Lemma 3, and if S is virtually polycyclic but not virtually nilpotent then we may take $V = \sqrt{S}$ (cf. [9, 15.1.6]).

Corollary 8. *If S is minimal TFNS then $h(\overline{S}) \geqslant 4$ and we may assume that $\overline{S}$ is perfect.*

Proof. If S is minimal TFNS then $\overline{S}$ is not solvable, since U is solvable. Hence $h(A) \geqslant 4$, by Lemma 1. After replacing S by S', if necessary, we may assume that $\overline{S}$ is perfect, by Lemma 6. □

We shall invoke the hypotheses and notation of Lemma 7 frequently in §5 and §8 below.

4. THE RELEVANT MINIMAL PERFECT GROUPS

A finite perfect group is *minimal* if all of its proper subgroups are solvable. The minimal simple groups are $PSL(2, q)$ for $q = 2^p$ with p a prime, $q = 3^\ell$ with ℓ odd, $q = p$ a prime such that $p^2 \equiv -1$ mod (5), a Suzuki group $Sz(2^p)$ with p an odd prime or $PSL(3, 3)$ [13]. Every finite simple group contains a minimal simple group [1]. (This is not entirely obvious from the definitions!) There appears to be no corresponding determination of minimal perfect groups.

We are interested in minimal perfect groups H which have non-trivial homomorphisms to $GL(n,\mathbb{Z})$, for some $n \leqslant 14$. We may in fact work with coefficients $\mathbb{Q}$, as every finite subgroup of $GL(n,\mathbb{Q})$ is conjugate into $GL(n,\mathbb{Z})$. Moreover, the rational group ring $\mathbb{Q}[H]$ is semisimple, which simplifies our analysis. The image of H in $GL(n,\mathbb{Q})$ is again a minimal perfect group, but may be a proper quotient of H. (When S is virtually nilpotent and $N=\sqrt{S}$, we may assume that $H=S/N$ embeds in $Aut(\mathbb{Q}N^{ab})$, by Lemma 3.)

There is a further simplification. The groups of interest to us which are not subgroups of $GL(10,\mathbb{Q})$ are subgroups of the symplectic group $Sp(12,\mathbb{Q})$. See §5 and Lemma 19 below.

Quite a lot can be done by hand. If p is a prime then the maximal power of p dividing the order of a finite subgroup $G<GL(n,\mathbb{Q})$ is $e_n(p)=\Sigma_{j\geqslant 0}\lfloor\frac{n}{p^j(p-1)}\rfloor$. In particular, $e_n(p)=0$ if $p>n+1$, while if $n<12$ then $e_n(7)\leqslant 1$ and $e_n(5)\leqslant 2$, and if $12\leqslant n\leqslant 14$ then $e_n(11)=e_n(13)=1$, $e_n(7)=2$ and $e_n(5)=3$. Moreover, if G is cyclic and of prime power order $p^k\geqslant 3$ then $n\geqslant p^{k-1}(p-1)$, while if G is cyclic of composite order not congruent to 2 mod (4) then n is bounded below by the corresponding sum. In the remaining case the minimal value of n is 1 less than this sum [5]. Thus the only prime powers of interest to us are 2^k with $k\leqslant 4$, 3, 5, 7, 9, 11 and 13.

The projective linear groups $PSL(2,q)$ contain cyclic subgroups of order $q-1$, and $q+1$, and so we may eliminate such groups with $q=2^k$ or 3^k and $k\geqslant 5$. The Suzuki group $Sz(2^p)$ has a cyclic subgroup of order 2^p+2^s+1, where $2s=p+1$, and so we may eliminate such groups with $p>3$. This leaves only $A_5=PSL(2,4)=PSL(2,5)$, $PSL(2,7)=SL(3,2)$, $SL(2,8)$, $PSL(2,13)$, $PSL(2,3^3)$, $Sz(8)$ and $PSL(3,3)$. The last four groups each have elements of order 13.

However we still need to consider extensions of such groups by solvable normal subgroups. At this point we shall simplify our task by using the findings of Lutowski and Szczepánski, who show that the minimal perfect groups with irreducible embeddings in $GL(n,\mathbb{Q})$ (for $n\leqslant 10$) are: A_5, $PSL(2,7)$, $SL(2,8)$, the universal central extensions $SL(2,5)$ and $SL(2,7)$, and $L_3(2)N2^3$, the non-split extension of $PSL(2,7)$ by $\mathbb{F}_2^3$ [7].

If H is simple then all non-trivial representations are faithful. We shall label the non-trivial $\mathbb{Q}$-irreducible $\mathbb{Q}$-rational characters of the groups of most interest to us by their degree. We shall also use the same symbols to denote the associated $\mathbb{Q}[H]$-modules.

A_5 has three: ρ_4,ρ_5,ρ_6.

$PSL(2,7)$ has four: $\tau_{6a},\tau_{6b},\tau_7,\tau_8$.

$SL(2,8)$ also has four: $\psi_7, \psi_8, \psi_{21}, \psi_{27}$.
The other three groups are not simple, but have faithful representations in dimensions 7 or 8.

$SL(2,5)$ has two: π_{8a} and π_{8b}.

$SL(2,7)$ has one: ξ_8.

$L_3(2)N2^3$ has two: λ_{7a} and λ_{7b}.

We shall also consider faithful representations which are reducible, but have no trivial summands. In particular, $SL(2,5)$ has two such representations in each of dimensions 12 and 13, given by $\pi_{8i} \oplus \widehat{\rho}_j$, where $i = a$ or b and $j = 4$ or 5, and $\widehat{\rho}_j$ is the representation which factors through ρ_j. Similarly, $L_3(2)N2^3$ has four such representations in dimension 13, but $SL(2,7)$ has none in dimensions < 14.

In the sections below on nilpotent groups we shall need to consider also *symplectic* representations. A representation ρ into $GL(2k,\mathbb{Q})$ is symplectic if it is conjugate into the subgroup $Sp(2k,\mathbb{Q})$.

Lemma 9. *Let $\rho : F \to GL(2k,\mathbb{Q})$ be an irreducible representation of a finite group F. Then ρ is symplectic if and only if the trivial representation $\mathbf{1}$ is a summand of the exterior square $\wedge_2\rho$.*

Proof. If ρ is symplectic then the associated skew-symmetric pairing defines a non-zero F-linear homomorphism from $\wedge_2\mathbb{Q}^n$ to $\mathbb{Q}$. Since $\mathbb{Q}[F]$ is semisimple, $\mathbf{1}$ is a summand of $\wedge_2\rho$.

Conversely, a projection from $\wedge_2\rho$ onto $\mathbf{1}$ gives a skew-symmetric pairing on $\wedge_2\mathbb{Q}^n$. Since ρ is irreducible the radical of this pairing is 0, and so the pairing is non-singular. Hence ρ is symplectic. ☐

The finite subgroups of $Sp(2k,\mathbb{Q})$ are determined in [4], for $2k \leqslant 12$. In particular, all such groups with order divisible by 13 are solvable, and the representation of $L_3(2)N2^3$ in $GL(8,\mathbb{Q})$ is not symplectic.

We shall need to know how the exterior squares of faithful representations of degree $\leqslant 10$ decompose as a sum of irreducible representations. This is an easy exercise in comparing characters. See [11, Chapter 2].

A_5: $\wedge_2\rho_4 \cong \rho_6$, $\wedge_2\rho_5 \cong \rho_4 \oplus \rho_6$, $\wedge_2\rho_6 \cong \rho_4 \oplus \rho_5 \oplus \rho_6$,
$\wedge_2 2\rho_4 \cong \mathbf{1} \oplus \rho_4 \oplus \rho_5 \oplus 3\rho_6$, $\wedge_2 2\rho_5 \cong \mathbf{1} \oplus 4\rho_4 \oplus 2\rho_5 \oplus 3\rho_6$,
$\wedge_2 2\rho_6 \cong 2\mathbf{1} \oplus 4\rho_4 \oplus 6\rho_5 \oplus 3\rho_6$, $\wedge_2(\rho_4 \oplus \rho_5) \cong 5\rho_4 \oplus \rho_5 \oplus 3\rho_6$,
$\wedge_2(\rho_4 \oplus \rho_6) \cong 3(\rho_4 \oplus \rho_5 \oplus \rho_6)$ and $\wedge_2(\rho_5 \oplus \rho_6) \cong 4\rho_4 \oplus 3\rho_5 \oplus 4\rho_6$.
$PSL(2,7)$: $\wedge_2\tau_{6a} \cong 2\mathbf{1} \oplus \tau_{6a} \oplus \tau_7$, $\wedge_2\tau_{6b} \cong 2\mathbf{1} \oplus \tau_{6b} \oplus \tau_7$,
$\wedge_2\tau_7 \cong 6\tau_{6a} \oplus \tau_7 \oplus \tau_8$ and $\wedge_2\tau_8 \cong 6\tau_{6a} \oplus 2\tau_7 \oplus \tau_8$
$SL(2,5)$: $\wedge_2\pi_{8a} \cong 3\mathbf{1} \oplus \widehat{\rho}_4 \oplus 3\widehat{\rho}_5 \oplus \widehat{\rho}_6$, and $\wedge_2\pi_{8b} \cong 6\mathbf{1} \oplus 4\widehat{\rho}_4 \oplus \widehat{\rho}_6$.
$SL(2,7)$: $\wedge_2\xi_8 \cong \mathbf{1} \oplus \widehat{\tau}_{6b} \oplus 2\widehat{\tau}_7 \oplus \widehat{\tau}_8$.
$SL(2,8)$: $\wedge_2\psi_7 \cong \psi_{21}$ and $\wedge_2\psi_8 \cong \psi_7 \oplus \psi_{21}$.
$L_3(2)N2^3$: $\wedge_2\lambda_{7a} \cong \lambda_{7a} \oplus \lambda_{14}$ and $\wedge_2\lambda_{7b} \cong \lambda_{7b} \oplus \lambda_{14}$.

One issue that complicates matters is that each irreducible $\mathbb{Q}[H]$-module $\mathbb{Q}A$ may derive from several distinct $\mathbb{Z}[H]$-modules. The simplest examples have $A \cong \mathbb{Z}^2$, $H = \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Q}A \cong \mathbf{1} \oplus sg$. There are two actions, one with $H^2(H;A) = \mathbb{Z}/2\mathbb{Z}$, realized by $G = \mathbb{Z} \rtimes_{-1} \mathbb{Z}$ and $\mathbb{Z} \times D_\infty$, and the other with $H^2(H;A) = 0$, realized by a non-split central extension of D_∞ by $\mathbb{Z}$. Thus $H^2(H;A)$ depends on more than just the $\mathbb{Q}[H]$-module $\mathbb{Q}A$.

The following information on torsion was provided by R. Lutowski. Let G be a crystallographic group with holonomy H. If $H \cong A_5$ and $h(G) = 4$ or 5 then G has 5-torsion. If $H \cong PSL(2,7)$ and $h(G) = 6$ or 7, or if $H \cong SL(2,7)$ and $h(G) = 8$, or if $H \cong L_3(2)N2^3$ and $h(G) = 7$ or 8 then G has 7-torsion. There are examples with $H \cong PSL(2,7)$ or $SL(2,8)$ and $h(G) = 8$ with no 7-torsion. (See also [3, Chapter 6] and §8 below.)

5. S virtually solvable but not virtually nilpotent

Let S be a finitely generated, infinite group which is torsion-free, virtually solvable and of finite Hirsch length. Then S has solvable normal subgroups $V \leqslant U$ such that S/U is crystallographic, U/V is finite, $V/I(V)$ has positive rank and U/V acts effectively on $V/I(V)$, by Lemma 7. Hence S/V is virtually abelian, and has a free abelian normal subgroup A of finite index. If U is nilpotent then we may assume $V = U$, and then $\sqrt{S/U}$ is the unique such subgroup of minimal index, but otherwise S/V may have a non-trivial torsion normal subgroup, and there may several such subgroups. Let W be the preimage of such a subgroup A in S, so $V < W$ and $W/V \cong A$.

Let A be one such free abelian normal subgroup of minimal index in S/V. Then $A \cong \mathbb{Z}^m$, where $m = h(S/V)$, and V^{ab} is a finitely generated $\mathbb{Z}[A]$-torsion module. The $\mathbb{Q}[A]$-module $M = \mathbb{Q} \otimes V^{ab}$ is a finite dimensional $\mathbb{Q}$-vector space, and $\mathbb{Q}[A]/Ann(M)$ is an Artinian ring. Let $Supp(M)$ be the set of maximal ideals $\mathfrak{m}$ in $\mathbb{Q}[A]$ which contain $Ann(M)$, and let $M_\mathfrak{m}$ be the submodule annihilated by a power of $\mathfrak{m}$. Then $M_\mathfrak{m}/\mathfrak{m}M_\mathfrak{m}$ is a non-trivial vector space over the field $\mathbb{K}_\mathfrak{m} = \mathbb{Q}[A]/\mathfrak{m}$, for each $\mathfrak{m} \in Supp(M)$, and $M = \oplus M_\mathfrak{m}$, where the summation is over $Supp(M)$.

The following lemma is a variation on Hall's criterion [9, 5.2.10].

Lemma 10. *Let S be a finitely generated, torsion-free group with subgroups U, V, W as above. Then S is virtually nilpotent if and only if U is virtually nilpotent and $Supp(M) = \{\mathfrak{e}\}$.*

Proof. If S is virtually nilpotent then $W/I(V)$ is also virtually nilpotent. Let Z_i be an ascending central series for a nilpotent subgroup of finite index in $W/I(V)$. The intersections $Z_i \cap (V/I(V))$ give rise to a filtration of M with subquotients annihilated by $\mathfrak{e}$.

Conversely, if $Supp(M) = \{\mathfrak{e}\}$ then M has such a filtration. This determines an ascending series $\{C_i \mid 0 \leqslant i \leqslant n\}$ for V such that $C_0 = I(V)$, each subquotient C_{i+1}/C_i is central in W/C_i and $C_n = V$. Hence $W/I(V)$ is nilpotent. Since W is torsion-free, it follows that if U is virtually nilpotent then W is virtually nilpotent, by a mild variation of Hall's criterion [9, 5.2.10]. □

We shall assume for the rest of this section that S is perfect and is minimal TFNS, and so $H = S/\widetilde{S}$ is a minimal simple group.

Lemma 11. *Let G be a finitely generated, perfect, virtually solvable subgroup of $GL(d,\mathbb{Q})$. If $d \leqslant 5$ then the order of $G/\widetilde{G}$ divides $2^{10}3^25$, and if $d \leqslant 3$ then $S = 1$.*

Proof. Since G is perfect, it is a subgroup of $SL(d,\mathbb{Q})$. Let m be the lowest common denominator for the entries of a generating set for G, and let $R = \mathbb{Z}[\frac{1}{m}]$. Then $G \leqslant SL(d,R)$. Let $p > m$ be prime, and let G_k be the kernel of the projection of G into $SL(d,R/p^kR)$. Then G/G_1 is a perfect subgroup of $SL(d,p)$, while G_k/G_{k+1} is an elementary p-group for all $k \geqslant 1$, and $\cap_{k\geqslant 1} G_k = 1$. In particular, either $G_1 = 1$, in which case G is finite, or G_1 maps onto $\mathbb{Z}/p\mathbb{Z}$, and so the maximal finite simple quotient of G has order dividing $|SL(d,p)|$.

Using the infinitude of primes in arithmetic progressions, we find that if $d = 5$ then the highest common factor of such orders (taken over all $p > m$) is $2^{10}3^25$. If $d = 3$ the highest common factor is 48, and so G has no simple quotient. Hence in this case $G = 1$. □

The argument extends to show that finitely generated perfect subgroups of $GL(n,\mathbb{K})$ have maximal simple quotients of order bounded as a function of n and $\mathbb{K}$, for any algebraic number field $\mathbb{K}$.

If $G < GL(5,\mathbb{Q})$ and $G/\widetilde{G}$ is a minimal simple group then $G/\widetilde{G} \cong A_5$ (which is a subgroup of $GL(4,\mathbb{Z})$). On the other hand, $PSL(2,7) < GL(\mathbb{Q},6)$. While $SL(2,8) < GL(7,\mathbb{Q})$, the arguments of Lemma 11 alone are not enough to decide whether an extension of $SL(2,8)$ by a solvable normal subgroup may embed in $GL(6,\mathbb{Q})$.

We shall sketch an argument. If such a group G with $G/\widetilde{G} \cong SL(2,8)$ has non-trivial image in $SL(6,p)$ for some $p > 7$ then it follows from the Aschbacher-Dynkin Theorem on subgroups of $GL(n,q)$ that the image is a central extension of $SL(2,8)$. Since $SL(2,8)$ is superperfect, central extensions split, and so $SL(2,8) < GL(6,p)$. If $p \equiv 1$ *mod* e, where $e = 126$ is the exponent of $SL(2,8)$ then representations of

$SL(2,8)$ over $\mathbb{F}_p$ lift to representations over $\mathbb{Q}(\zeta_e)$ of the same degree. (This uses Theorems 9.2.7 and 9.3.6 of [14].) Since the minimal non-trivial complex representation of $SL(2,8)$ has degree 7 this gives a contradiction.

Lemma 12. *If $\mathfrak{m}$ is fixed by S and $d = [\mathbb{K}_\mathfrak{m} : \mathbb{Q}] \leqslant 3$ then $\mathfrak{m} = \mathfrak{e}$.*

Proof. If $\mathfrak{m}$ is fixed by S then S acts on $\mathbb{K}_\mathfrak{m}$. The action is $\mathbb{Q}$-linear, and so factors through $GL(d,\mathbb{Q})$. If $d \leqslant 3$ this group has no non-trivial finitely generated perfect subgroups, by Lemma 11, and so the action is trivial. Hence $\mathfrak{m} = \mathfrak{e}$. □

Lemma 11 gives a sharper result when $S/\widetilde{S} \cong PSL(2,7)$.

Lemma 13. *If $\mathfrak{m}$ is not fixed by S then $h(S) \geqslant 9$. If $S/\widetilde{S} \cong A_5$ then $\dim_\mathbb{Q} M_\mathfrak{m} \leqslant 2$ and $[\mathbb{K}_\mathfrak{m} : \mathbb{Q}] \leqslant 2$. If $S/\widetilde{S} \cong PSL(2,7)$ then $M_\mathfrak{m} \cong \mathbb{Q}$ and $\mathbb{K}_\mathfrak{m} = \mathbb{Q}$. If $S/\widetilde{S} \cong SL(2,8)$ then S acts trivially on $Supp(M)$.*

Proof. If $S/\widetilde{S} \cong A_5$ then each non-trivial orbit $S\mathfrak{m}$ of the action of S on $Supp(M)$ must have at least 5 members, since S_4 has no nontrivial perfect subgroup. Hence $h(S) \geqslant 9$. Since $h(S/U) \geqslant 4$ we must have $|S\mathfrak{m}| \dim_\mathbb{Q} M_\mathfrak{m} \leqslant 10$ and so $\dim_\mathbb{Q} M_\mathfrak{m} \leqslant 2$. The largest proper subgroup of $PSL(2,7)$ has index 7, and so in this case each orbit has at least 7 members. If $S/\widetilde{S} \cong PSL(2,7)$ then $h(S/U) \geqslant 6$, so $|S\mathfrak{m}| \dim_\mathbb{Q} M_\mathfrak{m} \leqslant 8$ and $\dim_\mathbb{Q} M_\mathfrak{m} = 1$. Similarly, the largest proper subgroup of $SL(2,8)$ has index 9, but $h(S/U) \geqslant 8$, and so there is no non-trivial orbit. □

Together these lemmas give the following theorem.

Theorem 14. *Let S be TFNS but not virtually nilpotent, and let $H = S/\widetilde{S}$. Then $h(S) \geqslant 9$. If $H \cong PSL(2,7)$ then $h(S) \geqslant 13$, and if $H \cong SL(2,8)$ then $h(S) \geqslant 15$.*

Proof. Since S is not virtually nilpotent there is a $\mathfrak{m} \neq \mathfrak{e}$ in $Supp(M)$. If $H \cong A_5$ then $h(S/U) \geqslant 4$ and either $|S\mathfrak{m}| \geqslant 5$ or $[\mathbb{K}_\mathfrak{m} : \mathbb{Q}] \geqslant 4$, and so $h(S) \geqslant 8$. If $h(S) = 8$ then $h(S/U) = h(U) = 4$, and so S/U has an element of order 5, and V is abelian. Since U/V is finite, there is an element $s \in S$ whose image in S/V has order 5. Since S is torsion-free and V has rank 4 it follows that s must centralize V. But then the normal closure of s in S centralizes V also, and V is virtually nilpotent. Thus we must have $h(S) \geqslant 9$ if $H \cong A_5$.

Similarly, if $H \cong PSL(2,7)$ then $h(S/U) \geqslant 6$, and if $h(S/U) = 6$ or 7 then $h(U) \geqslant 7$. In this case we find that $h(S) \geqslant 13$.

If $H \cong SL(2,8)$ then $h(S/U) \geqslant 8$ and either $|S\mathfrak{m}| \geqslant 9$ or $[\mathbb{K}_\mathfrak{m} : \mathbb{Q}] \geqslant 7$, by the argument following Lemma 11. Hence $h(S) \geqslant 15$. (The torsion argument gives no extra leverage here.) □

There is an alternative argument for polycyclic groups, with a slightly sharper result.

Lemma 15. *Let S be virtually polycyclic. If $\mathbb{K}_\mathfrak{m} = \mathbb{Q}$ for all $\mathfrak{m} \in Supp(M)$ then S is virtually nilpotent.*

Proof. In this case we may assume that $V = \sqrt{S}$. Since S is virtually polycyclic the eigenvalues of the action of A on $\mathbb{Q} \otimes V^{ab}$ are algebraic integers. If $\mathbb{K}_\mathfrak{m} = \mathbb{Q}$ then the eigenvalues for the action on $M_\mathfrak{m}$ lie in $\mathbb{Q}$, and so must be ± 1. Hence the subgroup of S generated by $\sqrt{S}$ and all squares of elements of A is nilpotent. □

Theorem 16. *Let S be a virtually polycyclic group which is minimal TFNS, but not virtually nilpotent. Then $h(S) \geqslant 9$. If U is a normal subgroup which contains $\sqrt{S}$ and such that S/U is crystallographic then $h(S/U) \leqslant 6$ and so the holonomy of S/U is A_5 or $PSL(2,7)$.*

Proof. We may assume that U contains $\sqrt{S}$ as a subgroup of finite index, since $S/\sqrt{S}$ is virtually abelian. Then $h(\sqrt{S}) > h(S/\sqrt{S})$, by Lemma 5, and so $h(S) > 2h(S/U)$. Since S/U is perfect, $h(S/U) \geqslant 4$ and so $h(S) \geqslant 9$. If the holonomy of S/U is not A_5 or $PSL(2,7)$ then $h(S/U) \geqslant 8$. But then $h(S) \geqslant 17$, and so S is not minimal. □

6. NILPOTENT GROUPS

We shall say that a finitely generated nilpotent group N is *of type* $[m,n]$ if $N/\tilde{\gamma}_2 N \cong \mathbb{Z}^m$ and $\tilde{\gamma}_2 N/\tilde{\gamma}_3 N \cong \mathbb{Z}^n$. (Note that $n \leqslant \binom{m}{2}$, since $N/\tilde{\gamma}_3 N$ is a quotient of $F(m)$, the free group of rank m.)

Let $\mathbb{Q}N_{i/i+1} = \mathbb{Q} \otimes \tilde{\gamma}_i N/\tilde{\gamma}_{i+1} N$, for $i \geqslant 1$. Automorphisms of N must preserve the rational *commutator pairing*

$$[-,-]_\mathbb{Q} : \mathbb{Q}N^{ab} \wedge \mathbb{Q}N^{ab} \to \mathbb{Q}N_{2/3}.$$

This pairing has two related aspects. It is a skew-symmetric pairing on $\mathbb{Q}N^{ab}$, and also is an epimorphism of $\mathbb{Q}$-vector spaces. In the latter context we shall use the term *commutator epimorphism*. Let

$$R(N) = \{x \in \mathbb{Q}N^{ab} \mid [x,y]_\mathbb{Q} = 0, \ \forall\, y \in \mathbb{Q}N^{ab}\}$$

be the radical of the commutator pairing, and let $\overline{\mathbb{Q}N^{ab}} = \mathbb{Q}N^{ab}/R(N)$. Then $R(N) \cong \mathbb{Q} \otimes \zeta N/\tilde{\gamma}_3 N$.

Lemma 17. *Let N be a finitely generated nilpotent group such that $\tilde{\gamma}_3 N < \tilde{\gamma}_2 N$. Suppose that a nontrivial finite perfect group H acts effectively on N and fixes no nontrivial subspace of $\mathbb{Q}N^{ab}$. Then*

(1) $\dim_\mathbb{Q} \overline{\mathbb{Q}N^{ab}} \geq 4$ *and either* $R(N) = 0$ *or* $\dim_\mathbb{Q} R(N) \geqslant 4$*;*

(2) if there is a $\mathbb{Q}[H]$-linear epimorphism $\lambda : \mathbb{Q} \otimes \tilde{\gamma}_2 N/\tilde{\gamma}_3 N \to \mathbb{Q}$ then $\dim_{\mathbb{Q}} \overline{\mathbb{Q}N^{ab}} \geqslant 6$;

(3) $\mathbb{Q}N_{2/3}$ is a direct summand of $\mathbb{Q}N^{ab} \wedge \mathbb{Q}N^{ab}$.

Proof. The radical $R(N)$ is an H-invariant subspace of $\mathbb{Q}N^{ab}$. Since $\mathbb{Q}[H]$ is a semisimple ring, $R(N)$ has an H-invariant complement in $\mathbb{Q}N^{ab}$, which projects isomorphically onto $\overline{\mathbb{Q}N^{ab}}$. The complement is non-zero, since N is not virtually abelian.

Since finite subgroups of $GL(3, \mathbb{Q})$ are solvable, any H-invariant subspace of $\mathbb{Q}N^{ab}$ of dimension $\leqslant 3$ is fixed pointwise. Hence $\dim_{\mathbb{Q}} \overline{\mathbb{Q}N^{ab}} \geq 4$ and either $R(N) = 0$ or $\dim_{\mathbb{Q}} R(N) \geqslant 4$.

Suppose that $\lambda : \mathbb{Q} \otimes \tilde{\gamma}_2 N/\tilde{\gamma}_3 N \to \mathbb{Q}$ is a $\mathbb{Q}[H]$-linear epimorphism. The composite $\lambda \circ [-, -]_{\mathbb{Q}}$ is then a non-zero skew-symmetric pairing. Let $R(\lambda)$ be the radical of this pairing. The induced pairing on $V = \mathbb{Q}N^{ab}/R(\lambda)$ is nonsingular, and so has even dimension. Since $\mathbb{Q}[H]$ is semisimple and H fixes no non-trivial subspace of $\mathbb{Q}N^{ab}$, the action of H on V is nontrivial. Hence $\dim_{\mathbb{Q}} V \geqslant 6$, and either $R(\lambda) = 0$ or $\dim_{\mathbb{Q}} R(\lambda) \geqslant 4$. In particular, $\dim_{\mathbb{Q}} \mathbb{Q}N^{ab} \geqslant 6$.

The final assertion is clear, since $\mathbb{Q}[H]$ is semisimple and $[-, -]_{\mathbb{Q}}$ is an epimorphism. □

Lemma 18. *Let N be a finitely generated nilpotent group such that $\tilde{\gamma}_3 N < \tilde{\gamma}_2 N$. Suppose that a nontrivial finite perfect group H acts effectively on N and fixes no nontrivial subspace of $\mathbb{Q}N^{ab}$. If $h(N/\tilde{\gamma}_3 N) \leqslant 9$ then $[m, n] = [5, 4], [6, 1], [6, 2], [6, 3]$ or $[8, 1]$.*

Proof. The kernel of the commutator epimorphism has dimension $\binom{m}{2} - n$ and is H-invariant. Hence if N has type $[4, n]$ with $n = 3, 4$ or 5 then this kernel has dimension $\leqslant 3$, and so H acts trivially on it. Let ω be a nonzero 2-form in this kernel. Since 2-forms determine skew-symmetric pairings on the dual vector space, we may choose a basis $\{e_1, e_2, e_3, e_4\}$ for $\mathbb{Q}N^{ab}$, so that ω is one of $e_1 \wedge e_2$ or $e_1 \wedge e_2 + e_3 \wedge e_4$. If an automorphism of N fixes $e_1 \wedge e_2$ then it fixes the subspace of $\mathbb{Q} \otimes N^{ab}$ generated by e_1 and e_2. Hence we may assume that $\omega = e_1 \wedge e_2 + e_3 \wedge e_4$, and that H does not fix any 2-dimensional subspace of $\mathbb{Q}N^{ab}$. In this case the skew-symmetric pairing determined by ω is non-degenerate and H acts symplectically.

If $h(N/\tilde{\gamma}_3 N) \leqslant 7$ then $\dim_{\mathbb{Q}} \overline{\mathbb{Q}N^{ab}} \geqslant 4$, and so $\dim_{\mathbb{Q}} \mathbb{Q} \otimes \tilde{\gamma}_2 N/\tilde{\gamma}_3 N \leqslant 3$. Hence H acts trivially on $\mathbb{Q} \otimes \tilde{\gamma}_2 N/\tilde{\gamma}_3 N$, and so N is of type $[6, 1]$, by (2). Similar arguments show that if N is of type $[m, n]$ with $n \leqslant 3$ then $m \geqslant 5 + n$, and if $m > 6$ then $m \geqslant 8$.

Thus we are left with only types [5,4], [6,1], [6,2], [6,3] and [8,1]. □

We shall show below that [5,4] and [6,3] are not possible. (In fact we shall see that there is no minimal TFNS group S with $h(S) \leqslant 10$.)

7. S virtually nilpotent

In this section we shall assume that S is virtually nilpotent, but not virtually abelian, and that S is perfect, $h(S) \leqslant 14$ and $H = S/\sqrt{S}$ is a minimal perfect group. However we do not assume here that H is simple, and we do not need the notation of Lemma 7, as we may take $V = U = I(\sqrt{S})$, by Lemma 3. Our goal is to limit the possibilities for H and for the type of $N = \sqrt{S}$. The fact that S is torsion-free is used only through Lemma 3, to show that H embeds in $Aut(\mathbb{Q}N^{ab})$.

We shall play off the $\mathbb{Q}[H]$-module structures of $\mathbb{Q}N^{ab}$ and $\mathbb{Q}N_{2/3} = \mathbb{Q} \otimes \tilde{\gamma}_2 N/\tilde{\gamma}_3 N$ against each other. The key conditions are

(1) $\dim_{\mathbb{Q}} N^{ab} + \dim_{\mathbb{Q}} \mathbb{Q}N_{2/3} \leqslant h(S) \leqslant 14$ and $\dim_{\mathbb{Q}} N_{2/3} \geqslant 1$;
(2) $\mathbb{Q}N_{2/3}$ is a $\mathbb{Q}[H]$-summand of $\wedge_2 \mathbb{Q}N^{ab}$;
(3) $\mathbb{Q}$ is not a $\mathbb{Q}[H]$-summand of $\mathbb{Q}N^{ab}$, by Lemmas 2 and 3;
(4) if $\mathbb{Q}$ is a $\mathbb{Q}[H]$-summand of $\mathbb{Q}N_{2/3}$ then $\mathbb{Q}N^{ab}$ has a symplectic summand, by Lemma 9.

Let $[m, n]$ be the type of N. Then $m + n \leqslant h(S) \leqslant 14$ and $n > 0$, by (1). If $\mathbb{Q}$ is a $\mathbb{Q}[H]$-summand of $\mathbb{Q}N_{2/3}$ then $\mathbb{Q}N^{ab} \cong V_0 \oplus V_1$, where V_1 supports a nonsingular skew-symmetric pairing, with even rank r. Since finite subgroups of $Sp(4, \mathbb{Q})$ are solvable, $r \geqslant 6$, and since H acts effectively on V_0, either $V_0 = 0$ or $m - r = \dim_{\mathbb{Q}} V_0 \geqslant 4$. In the latter case $m \geqslant 10$. There is always such a summand $\mathbb{Q}$ if $n \leqslant 3$.

On the other hand, the fact that A_5 acts effectively on $F(4)/\gamma_3(F(4))$ shows that $\mathbb{Q}N^{ab}$ need not have a symplectic summand.

Lemma 19. *If $H = S/\sqrt{S}$ is not a subgroup of $GL(10, \mathbb{Q})$ then it is a subgroup of $Sp(12, \mathbb{Q})$.*

Proof. We must have $11 \leqslant m \leqslant 13$, and so $n \leqslant 3$. Hence $\mathbb{Q}N_{2/3}$ is a trivial H-module. Let $\lambda : \mathbb{Q}N_{2/3} \to \mathbb{Q}$ be an epimorphism. Then H preserves the non-zero skew-symmetric pairing $\omega = \lambda \circ [-,-]_{\mathbb{Q}}$, and $\mathbb{Q}N^{ab} \cong R(\omega) \oplus V$, where the induced pairing on V is non-singular, and so has even dimension. If $R(\omega) = 0$ then $m = 12$ and H is a subgroup of $Sp(12, \mathbb{Q})$, since $\mathbb{Q}N^{ab}$ is a faithful H-module. Otherwise $r = \dim_{\mathbb{Q}} R(\omega) \geqslant 4$ and $d = \dim_{\mathbb{Q}} V = 6$ or 8, since $\mathbb{Q}N^{ab}$ has no trivial summands. Let π_R and π_V be the projections of H into $GL(r, \mathbb{Q})$ and $Sp(d, \mathbb{Q})$, respectively. Then $K_R = \mathrm{Ker}(\pi_R)$ and $K_V = \mathrm{Ker}(\pi_V) = 1$ are each subgroups of $\widetilde{H}$, the solvable radical of H, and $K_R \cap K_V = 1$. Since $d \leqslant 8$ and $\pi_V(H)$ is perfect, $\pi_V(H)$ is either A_4, $PSL(2, 7)$ or $SL(2, 5)$ [4]. In the first two cases $K_V = \widetilde{H}$, and so $K_R \leqslant K_V$. Hence π_R

is injective. Otherwise, π_V maps K_R injectively to $\mathbb{Z}/2\mathbb{Z}$, the solvable radical of $SL(2,5)$. Hence $H \cong SL(2,5)$ and π_V is injective. Since $r < 10$ and $d < 12$, this proves the lemma. □

The only groups with representations satisfying conditions (1)–(4) above are $H \cong A_5$, $PSL(2,7)$, $SL(2,5)$, $SL(2,7)$ or $L_3(2)N2^3$. In particular, this observation together with Theorem 14 shows that there is no virtually solvable, minimal TFNS group S with $S/\widetilde{S} \cong SL(2,8)$ and $h(S) \leqslant 14$.

Consideration of the decompositions of $\mathbb{Q}N^{ab}$ and $\mathbb{Q}N_{2/3}$ as $\mathbb{Q}[H]$-modules shows that the remaining possibilities for $[m,n]$ and H are:

$m = 4$ and $n = 6$. Only $H = A_5$.
$m = 5$ and $n = 4$ or 6. Only A_5.
$m = 6$ and $n = 1$ or 2. Only $PSL(2,7)$.
$m = 6$ and $n = 4$ or 5. Only A_5.
$m = 6$ and $n = 6$. A_5 or $PSL(2,7)$.
$m = 6$ and $n = 7$ or 8. Only $PSL(2,7)$.
$m = 7$ and $n = 6$. Only $PSL(2,7)$.
$m = 7$ and $n = 7$. $PSL(2,7)$ or $L_3(2)N2^3$.
$m = 8$ and $n = 1$. A_5, $SL(2,5)$ or $SL(2,7)$.
$m = 8$ and $n = 2$ or 3. Only $SL(2,5)$.
$m = 8$ and $n = 4$ or 5. A_5 or $SL(2,5)$.
$m = 8$ and $n = 6$. $A_5, PSL(2,7)$ or $SL(2,5)$.
$m = 9$ and $n = 4$ or 5. Only A_5.
$m = 10$ and $n = 1$. Only A_5 and $\mathbb{Q}N^{ab} \cong \rho_5 \oplus \rho_5$.
$m = 10$ and $n = 4$. Only A_5.
$m = 12$ and $n = 1$ or 2. $A_5, PSL(2,7)$ or $SL(2,5)$.
$m = 13$ and $n = 1$. A_5, $PSL(2,7)$ or $SL(2,5)$.

Parallel arguments apply further down the $\mathbb{Q}$-lower central series, since conjugation in S induces actions of $H = S/\sqrt{S}$ on each of the subquotients $\tilde{\gamma}_i N/\tilde{\gamma}_{i+1}N$, and there are natural epimorphisms from $\mathbb{Q}N^{ab} \otimes \mathbb{Q}N_{i/i+1}$ to $\mathbb{Q}N_{i+1/i+2}$, for all $i \geqslant 1$ [9, 5.2.5].

Theorem 20. *If S is minimal TFNS, virtually nilpotent and $h(S) \leqslant 14$ then either $\tilde{\gamma}_4\sqrt{S} = 1$, or $S/\sqrt{S} \cong A_5$, $[m,n] = [8,1]$, $h(\tilde{\gamma}_3\sqrt{S}) = 5$ and $\tilde{\gamma}_4\sqrt{S} \cong \mathbb{Z}$. In either case, $\sqrt{S}$ is metabelian.*

Proof. Let $N = \sqrt{S}$ have type $[m,n]$ and let $H = S/N$.

If $\tilde{\gamma}_3 N = 1$ then $I(N)$ is central in N, and so N is metabelian. If $\tilde{\gamma}_3 N \neq 1$ then $\mathbb{Q}N_{3/4}$ must be a summand of $\mathbb{Q}N^{ab} \otimes \mathbb{Q}N_{2/3}$, of rank $\leqslant 14 - m - n$. Checking the possibilities, we see that either $H \cong A_5$ and $[m,n] = [4,6]$, [5,4], [5,5], [6,4], [6,6], [8,1], [8,4] or [9,4], or $H \cong PSL(2,7)$ and $[m,n] = [6,1]$, [6,2] or [6,6].

If $\tilde{\gamma}_4 N = 1$ then N is metabelian, since $G'' \leqslant \gamma_4 G$ for any group G. If $\tilde{\gamma}_4 N \neq 1$ then $\mathbb{Q}N_{4/5}$ must be a summand of $\mathbb{Q}N^{ab} \otimes \mathbb{Q}N_{3/4}$. Since $h(S) \leqslant 14$ we must have $H \cong A_5$, $[m,n] = [8,1]$, $\mathbb{Q}N_{3/4} = \rho_4$ and $\mathbb{Q}N_{4/5} = \mathbf{1}$. Hence $h(\tilde{\gamma}_3\sqrt{S}) = 5$ and $\tilde{\gamma}_4\sqrt{S} \cong \mathbb{Z}$. Closer inspection shows that since $\tilde{\gamma}_2 N/\tilde{\gamma}_3 N$ is cyclic, $I(N)' \leqslant \tilde{\gamma}_5 N = 1$, and so N is again metabelian. □

8. Torsion in the crystallographic quotients

In this section we shall use the fact that the crystallographic quotients of our groups often have "large" finite subgroups to reduce the list of unsettled cases further.

Lemma 21. *Let S be a finitely generated, perfect, torsion-free group with a normal subgroup N such that $H = S/N$ is finite and $I(N) \cong \mathbb{Z}^n$. If $S/I(N)$ has an element of prime order $p > n$ whose image in H normally generates H then $I(N) \leqslant \zeta S$.*

Proof. Let $s \in S$ be an element whose image in $S/I(N)$ has order p. Then s normalizes $I(N)$. Since the subgroup generated by $I(N)$ and s is torsion-free and $p > n$, this is only possible if s centralizes $I(N)$. Since $I(N)$ is normal in S, the normal closure of s centralizes $I(N)$, and so $I(N) \leqslant \zeta S$. □

With this lemma we may exclude the cases with $H = A_5$, and $[m,n] = [5,4]$, [8,4] or [9,4] (for which $p = 5$), and those with $H \cong PSL(2,7)$ and $[m,n] = [6,6]$ or $[7,6]$ (for which $p = 7$). Hence $\tilde{\gamma}_3\sqrt{S} = 1$ in all cases excepting when $H \cong A_5$ and $[m,n] = [4,6]$, [6,4], [6,5], [6,6] or[8,1], or $H \cong PSL(2,7)$ and $[m,n] = [6,1]$ or [6,2]. Similarly, if $H \cong PSL(2,7)$ and $[m,n] = [6,6]$ then $\tilde{\gamma}_3\sqrt{S} \neq 1$.

Let H be a finite group which acts effectively on an abelian group A and let G be an extension of H by A corresponding to $\xi \in H^2(H;A)$. If $J < H$ has order relatively prime to that of $H^2(H;A)$ then the restriction to J of $c(\xi)$ is 0, and so the restricted extension splits. If J is non-abelian it then follows that no extension of G by an abelian normal subgroup of rank $\leqslant 3$ can be torsion-free.

Lemma 22. *Let G be a crystallographic group with translation subgroup A and holonomy H. Suppose that the Sylow p-subgroup of H is a cyclic subgroup C which is properly contained in its normalizer $N_H(C)$ and acts without fixed points on A. Then G has a subgroup isomorphic to $N_H(C)$.*

Proof. Let $D = N_H(C)/C$. Then $H^1(D; H^1(C;A)) = 0$, since $H^1(C;A)$ is a finite p-group and the order of D is prime to p. Since C is cyclic and acts on A without fixed points, $H^2(C;A) = H^0(C;A) = 0$. The LHS Spectral sequence gives $H^2(N_H(C);A) = 0$, and so the projection of G onto H splits over the subgroup $N_H(C)$. □

Now suppose that S is minimal TFNS and has a metabelian normal subgroup N such that $G = S/I(N)$ is crystallographic with holonomy $H = S/N$, and that H has a cyclic Sylow p-subgroup C which is properly contained in its normalizer $N_H(C)$, for some prime p. If $h(S/I(N)) = p-1$ then elements of H of order p act without fixed points. The preimage in S of a finite subgroup $F < G$ is a Bieberbach group with holonomy F.

The Sylow 5-subgroups of A_5 are cyclic and have normalizer D_{10}. No 5-dimensional Bieberbach group has holonomy D_{10} [2], and so there are no torsion-free extensions of D_{10} by an abelian group of rank 5. Taking into account Theorem 14, we may conclude that $h(S) \geqslant 10$.

If S is virtually nilpotent, $N = \sqrt{S}$, $H \cong A_5$ and $h(N/I(N)) = 8$ then $\mathbb{Q}N^{ab} = \rho_4 \oplus \rho_4$. Let C be a a Sylow 5-subgroup of A_5. As a $\mathbb{Z}[C]$-module $N/I(N)$ is a direct sum $L \oplus L'$, where L and L' are irreducible and of rank 4 as abelian groups. Hence they are each either the augmentation ideal in $\mathbb{Z}[C]$ or its $\mathbb{Z}$-linear dual. In either case, C acts on $N/I(N)$ without fixed points, and so $D_{10} < S/I(N)$. Therefore we cannot have $h(I(N)) \leqslant 5$. Hence we may also exclude the cases with $S/\sqrt{S} \cong A_5$ and $m = 8$, excepting perhaps when $h(S) = 14$ and $[m,n] = [8,1]$ or [8,6]. However the case $[m,n] = [8,1]$ follows on first using the theorem to show that $\overline{S} = S/\tilde{\gamma}_3\sqrt{S}$ has a subgroup isomorphic to D_{10}.

If G is crystallographic, $h(G) = 6$, $H \cong A_5$ and $H^2(H;A) \cong \mathbb{Z}/5\mathbb{Z}$ then the projection onto H splits over a subgroup isomorphic to A_4. However this gives us nothing new, as A_4 is the holonomy of a 4-dimensional Bieberbach group.

The faithful 8-dimensional representations of $SL(2,5)$ each restrict to fixed-point free representations of the Sylow 5-subgroup, and so corresponding crystallographic groups each have subgroups isomorphic to the normalizer of this subgroup, which is metacyclic of order 20. (This is the Borel subgroup of $SL(2,5)$.) This is not the holonomy group of a 6-dimensional Bieberbach group [2], and so we may exclude the cases with $S/\sqrt{S} \cong SL(2,5)$ and $m = 8$. Similarly for the faithful 12-dimensional representations of $SL(2,5)$, and for the 12-dimensional representations of A_5 with character $3\rho_4$.

If $S/\sqrt{S} \cong SL(2,5)$ and $N = \sqrt{S}$ is of type [13,1] then $\mathbb{Q}N^{ab} \cong \pi_{8i} \oplus \hat{\rho}_5$, for some $i = a$ or b, and $SL(2,5)$ acts symplectically. Hence $h(\zeta N) = 6$. Let $\overline{N}$ be the quotient of $N/\zeta N$ by its torsion subgroup.

Then $\mathbb{Q}\overline{N} \cong \pi_{8i}$, and so $SL(2,5)$ acts effectively on $\overline{N}$. As before, $S/\zeta N$ must have a subgroup isomorphic to the Borel subgroup of $SL(2,5)$. Since S is torsion-free, we may exclude this case also.

The lemma also applies when $h(S/I(N)) = 6$, $H = PSL(2,7)$ and $p = 7$, with $D = M_{7,3}$, the metacyclic group of order 21. (This is the image of the Borel subgroup of $SL(2,7)$.) R. Lutowski has used CARAT to verify that $M_{7,3}$ is not the group of any 8-dimensional Bieberbach group [6]. This shall enable us to substantially reduce the role of $PSL(2,7)$ in answering our question.

Theorem 23. *Let S be a torsion-free virtually nilpotent group such that $S/\sqrt{S} \cong PSL(2,7)$, $h(S/I(\sqrt{S}) = 6$, and $\tilde{\gamma}_3\sqrt{S} = 1$. Then $h(S) \geqslant 15$.*

Proof. The group $S/I(\sqrt{S})$ has a subgroup $L \cong M_{7,3}$, by Lemma 22. Since L acts effectively on $I(\sqrt{S})$ and S is torsion-free, the preimage in S of L is a Bieberbach group. Since $M_{7,3}$ is not the group of any 8-dimensional Bieberbach group, $h(W) \geqslant 9$, and so $h(S) \geqslant 15$. □

In particular, if $S/\sqrt{S} \cong PSL(2,7)$ then $[m,n] \neq [6,1]$, [6,2], [6,6], [6,7], or [6,8]. An argument parallel to the one above for the case with $H \cong A_5$ and $h(S/I(\sqrt{S})) = 8$ shows that we may extend Theorem 23 to exclude the cases with $S/\sqrt{S} \cong PSL(2,7)$ and $[m,n] = [12,1]$, [12,2] or [13,1]. (If $N = \sqrt{S}$ and $m = 12$ then $\mathbb{Q}N^{ab} = \tau_{6i} \oplus \tau_{6j}$ for some $i, j \in \{a, b\}$. If $m = 13$ then $\mathbb{Q}N^{ab} = \tau_{6k} \oplus \tau_7$ for some $k = a$ or b and then $\zeta N \cong \mathbb{Z}^8$ and $h(S/\zeta N) = 6$.)

Theorem 23 may be extended to the case when S is not virtually nilpotent, provided that (in the notation of Lemma 7) the normal subgroup V is abelian. We shall use the result of Lutowski [6] again, together with the simpler observation that 9 is the smallest dimension of a Bieberbach group with holonomy cyclic of order 21 [5].

Theorem 24. *Let S be a minimal TFNS group, and suppose that S has normal subgroups $V < U < S$ such that S/U is crystallographic, with $h(S/U) = 6$ and holonomy $PSL(2,7)$, U/V is a finite solvable group and V is abelian. Then $h(V) \geqslant 9$ and $h(S) \geqslant 14$.*

Proof. We may assume without loss of generality that S is perfect. Suppose that $h = h(V) < 9$. The group S/U has a subgroup isomorphic to $M_{7,3}$, by Lemma 22. Let W be the preimage of this subgroup in S. Then W/V is a finite solvable group. It acts effectively on V, for otherwise some element of W with non-trivial image in U would centralize V. But any such element must map non-trivially to the simple quotient $PSL(2,7)$, and it would follow that V must be central in S. In particular, S would be virtually nilpotent, and the action of the holonomy on V would be trivial. Hence $U = V \cong \mathbb{Z}$ or $\mathbb{Z}^2$, since $h(S/U) = 6$ and S/U has holonomy $PSL(2,7)$. But no extension of a finite nonabelian group by $\mathbb{Z}$ or $\mathbb{Z}^2$ is torsion-free. Hence we may assume that W/V embeds in $GL(8,\mathbb{Q})$.

Finite subgroups of $GL(8,\mathbb{Q})$ have order dividing $2^{15}3^5 5.7$. Since W/V maps onto $M_{7,3}$ it has order $3^k 7q$, where $1 \leqslant k \leqslant 5$ and $(q,21) = 1$. Since W/V is solvable it has a Hall $\{3,7\}$-subgroup K, of order $3^k 7$ [9, 9.1.7]. The Sylow 7-subgroup Syl_7K is normal in K, since 1 is the only divisor of 3^k which is congruent to 1 *mod* (7). Since K/Syl_7K is a 3-group it has cyclic subgroups of order 3, and the preimage in K of such a subgroup is a subgroup M of order 21.

Since M is finite it preserves a lattice $L \leqslant V$, and so W has a finitely generated subgroup which is an extension of M by $L \cong \mathbb{Z}^h$. If W were torsion-free then this subgroup would be a Bieberbach group of dimension $h \leqslant 8$, and with holonomy of order 21. But there are no such Bieberbach groups. Hence we must have $h(V) \geqslant 9$. □

Theorems 14 and 16 leave open the possibility of a minimal TFNS group S which is not virtually nilpotent, and with $S/\widetilde{S} \cong PSL(2,7)$, $h(V) = 8$ and $h(I(V)) = 1$. We shall exclude this possibility now. If S is such a group then it has a subgroup W which contains V as a normal subgroup of index 21, and $M = W/V$ acts effectively on $V/I(V)$. Since $h(I(V)) = 1$, the group V is nilpotent, and M preserves the commutator epimorphism. Hence M acts symplectically. Finite subgroups of $Sp(4,\mathbb{Q})$ do not have order divisible by 7, and so $h(V/\zeta V) = 6$. Since M acts effectively on $\mathbb{Q} \otimes V/\zeta V$ it cannot be cyclic of order 21, and so is metacyclic. It then follows from Lemma 22 that the projection of $W/\zeta V$ onto M splits, and so W contains a subgroup which is an extension of M by an abelian group of rank 2. This contradicts the assumption that S is torsion-free.

Every 8-dimensional crystallographic group with holonomy $SL(2,7)$ is a semidirect product $\mathbb{Z}^8 \rtimes_\theta SL(2,7)$, for some effective action θ [3, page 295], and so we may exclude the case with $S/\sqrt{S} \cong SL(2,7)$ and $[m,n] = [8,1]$. Similarly, the cohomology classes corresponding to extensions of $L_3(2)N2^3$ by $\mathbb{Z}^8$ which are crystallographic have order $\leqslant 2$ [3, page 298]. Hence such extensions split over subgroups of $L_3(2)N2^3$ of odd order, Since $M_{7,3}$ is such a subgroup, and is not the holonomy of a 7-dimensional Bieberbach group, we may exclude the case with $S/\sqrt{S} \cong L_3(2)N2^3$ and $[m,n] = [7,7]$.

In the light of the above arguments we find that $h(S) \geqslant 10$, and if S is not virtually nilpotent then $S/\widetilde{S} \cong A_5$. If S is virtually nilpotent then the list of possibilities for $[m,n]$ and H reduces to

$[m,n]=[4,6]$. $H\cong A_5$ and $\mathbb{Q}\sqrt{S}_{3/4}=\rho_4$ or 0.
$[m,n]=[5,6]$. $H\cong A_5$.
$[m,n]=[6,4]$. $H\cong A_5$ and $\mathbb{Q}\sqrt{S}_{3/4}=\rho_4$ or 0.
$[m,n]=[6,5]$. $H\cong A_5$.
$[m,n]=[6,6]$. $H\cong A_5$ and $\mathbb{Q}\sqrt{S}_{3/4}=\mathbf{2}$ or $\mathbf{1}$.
$[m,n]=[7,7]$. $H\cong PSL(2,7)$.
$[m,n]=[8,6]$. $H\cong A_5$.
$[m,n]=[8,6]$. $H\cong PSL(2,7)$.
$[m,n]=[9,5]$. $H\cong A_5$.
$[m,n]=[10,1]$. $H\cong A_5$ and $\mathbb{Q}N^{ab}\cong\rho_5\oplus\rho_5$.
$[m,n]=[10,4]$. $H\cong A_5$.
$[m,n]=[12,1]$. $H\cong A_5$, $\mathbb{Q}N^{ab}\cong\rho_6\oplus\rho_6$ and $\mathbb{Q}\sqrt{S}_{3/4}=\mathbf{1}$ or 0.
$[m,n]=[12,2]$. $H\cong A_5$ and $\mathbb{Q}N^{ab}\cong\rho_6\oplus\rho_6$.
$[m,n]=[13,1]$. $H\cong A_5$.

In particular, if $h(S)<14$ then $H=S/\sqrt{S}\cong A_5$, and $\tilde\gamma_3\sqrt{S}=1$ in all cases except for $[m,n]=[4,6]$, [6,4], [6,6] or [12,1].

We cannot exclude type $[4,6]$ by arguments involving just the lower central series. Let $\{w,x,y,z\}$ be a basis for $F(4)$, and define endomorphisms σ and τ by $\sigma(w)=w^{-1}$, $\sigma(x)=wxy$, $\sigma(y)=y^{-1}$, $\sigma(z)=yz$ and $\tau(w)=x^{-1}$, $\tau(x)=xyz$, $\tau(y)=z^{-1}$ and $\tau(z)=(wxy)^{-1}$. These are automorphisms, since $\sigma^2=\tau^3=(\sigma\tau)^5=1$, and define a monomorphism $\theta:A_5\to Aut(F(4))$, with rational abelianization ρ_4 in $GL(4,\mathbb{Q})$. Let S be an extension of A_5 by $N=F(4)/\gamma_3F(4)$, with action θ. Then $\sqrt{S}=F(4)/\gamma_3F(4)$ is of type [4,6].

If $\alpha:H\to Aut(N)$ is a homomorphism then the semidirect product $N\rtimes_\alpha H$ is a basepoint for the set of extensions of H by N with outer action corresponding to α, and so determines a natural bijection from $H^2(H;\zeta N)$ to the set of such extensions. The restriction of an extension ξ to a subgroup $J<H$ splits if and only if the corresponding cohomology class $c(\xi)$ restricts to 0 in $H^2(J;\zeta N^{\alpha|_J})$. (This is not clear if the outer action does not factor though $Aut(N)$!)

We may apply this observation to S and to $J=A_4<A_5$. Since $H^2(A_5;\zeta\sqrt{S})$ has order 5 [3, page 273] and $(|A_4|,5)=1$, the preimage of J in S is a semidirect product, and so S has torsion. Taking into account the Jacobi identities, we see that $\mathbb{Q}F(4)_{3/4}\cong\rho_4\oplus2\rho_5\oplus\rho_6$, and so $F(4)$ has a canonical θ-invariant subgroup K such that $\gamma_4F(4)<K<\gamma_3F(4)$ and $h(F(4)/K)=14$. A similar argument then shows that any extension of A_5 by $F(4)/K$ with outer action induced by θ must have 2-torsion. However we do not know whether such arguments apply to other virtually nilpotent groups S with $\sqrt{S}$ of type [4,6].

9. SOME QUESTIONS

1) Is there a minimal TFNS group which is an extension of a crystallographic group by $\mathbb{Z}$? In particular, does every perfect 12-dimensional crystallographic group with holonomy A_5 have either D_{10} or A_4 as a subgroup?

2) If H is the holonomy group of an n-dimensional infranilmanifold is it also the holonomy group of a flat n-manifold? This is so if $n \leqslant 4$, by inspection of the known groups. In general, H is the holonomy of a crystallographic group in dimension $\leqslant n$, by Lemma 3.

(The converse fails for $n = 4$, since A_4 is the holonomy group of a flat 4-manifold, but not of any other 4-dimensional infranilmanifold.)

3) If S is a minimal TFNS group must it be virtually polycyclic?

4) If S is a minimal TFNS group must it be perfect?

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